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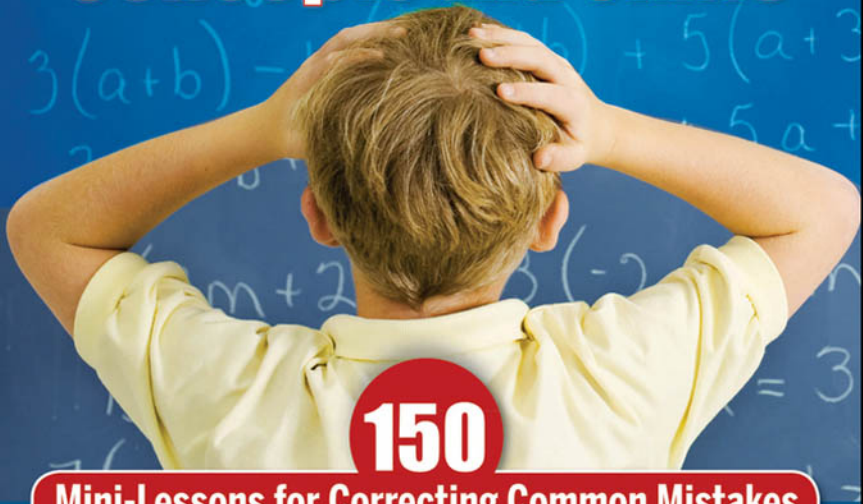
THE Algebra Teacher's Guide to Reteaching Essential Concepts and Skills



150 Mini-Lessons for Correcting Common Mistakes

Judith A. Muschla • Gary Robert Muschla • Erin Muschla

The Algebra Teacher's Guide to Reteaching Essential Concepts and Skills

150 MINI-LESSONS FOR CORRECTING COMMON MISTAKES

Judith A. Muschla
Gary Robert Muschla
Erin Muschla

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ABOUT THIS BOOK

ALGEBRA IS THE BRIDGE between basic and higher mathematics. Studying algebra sharpens students' overall proficiency in math, develops problem-solving strategies and skills, and fosters the ability to recognize, analyze, and express mathematical relationships. Students who master algebra usually go on to be successful in higher mathematics such as geometry, trigonometry, and calculus.

The Algebra Teacher's Guide to Reteaching Essential Concepts and Skills consists of 150 mini-lessons divided into eight sections:

- **Section 1:** Integers, Variables, and Expressions
- **Section 2:** Rational Numbers
- **Section 3:** Equations and Inequalities
- **Section 4:** Graphs of Points and Lines
- **Section 5:** Monomials and Polynomials
- **Section 6:** Rational Expressions
- **Section 7:** Irrational and Complex Numbers
- **Section 8:** Functions

The mini-lessons presented in the sections are based on a general algebra curriculum. Many of the mini-lessons in Sections 1 and 2 focus on prerequisite skills that students must master if they are to succeed in algebra.

Each mini-lesson, consisting of teaching notes and a reproducible worksheet, concentrates on a specific algebraic concept or skill students often have trouble mastering. Each mini-lesson requires only a few minutes to deliver and can be used with individual students, groups, or the whole class.

The teaching notes provide background information on the topic and suggestions for instruction. Each includes an "extra help" statement that you may share with your students about the topic of the mini-lesson. Answer keys are included at the end of each mini-lesson, making it easy for you to check your students' answers to the problems on the worksheets.

The reproducible worksheets provide your students with additional practice, helping them to master the concept or skill on which the mini-lesson focuses. The typical worksheet contains information for students, examples, and problems, culminating with a "challenge" problem that requires higher-level thinking. For these problems, students must demonstrate their understanding of the material by identifying faulty reasoning, explaining a process, or correcting a procedure. You may assign any or all of the problems, depending on the needs of your students.

Because each worksheet is set on one page to make photocopying easy, your students will likely need to work out the problems on another sheet of paper.

The worksheets can serve a variety of purposes:

- Remediation to master material
- Reinforcement of learned material
- Closure of the day's topic
- Review of the previous day's work
- Sponge activities to fill transitional times (for example, when some students complete class work sooner than others)

We hope that these mini-lessons and worksheets will enable you to help your students achieve proficiency in algebra, firming the foundation for their continued progress in math. Our best wishes to you for a successful and enjoyable year.

October 2011

Judith A. Muschla
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Erin Muschla
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ABOUT THE AUTHORS

Judith A. Muschla received her B.A. in mathematics from Douglass College at Rutgers University and is certified to teach K–12. She taught mathematics in South River, New Jersey, for over twenty-five years at various levels at South River High School and South River Middle School. As a team leader at the middle school, she wrote several math curricula, coordinated interdisciplinary units, and conducted mathematics workshops for teachers and parents. She has also served as a member of the state review panel for New Jersey's Mathematics Core Curriculum Content Standards.

Together, Judith and Gary Muschla have coauthored several math books published by Jossey-Bass: *Hands-on Math Projects with Real-Life Applications, Grades 3–5* (2009); *The Math Teacher's Problem-a-Day, Grades 4–8* (2008); *Hands-on Math Projects with Real-Life Applications, Grades 6–12* (1996; second edition, 2006); *The Math Teacher's Book of Lists* (1995; second edition, 2005); *Math Games: 180 Reproducible Activities to Motivate, Excite, and Challenge Students, Grades 6–12* (2004); *Algebra Teacher's Activities Kit* (2003); *Math Smart! Over 220 Ready-to-Use Activities to Motivate and Challenge Students, Grades 6–12* (2002); *Geometry Teacher's Activities Kit* (2000); and *Math Starters! 5- to 10-Minute Activities to Make Kids Think, Grades 6–12* (1999).

Gary Robert Muschla received his B.A. and M.A.T. from Trenton State College and taught in Spotswood, New Jersey, for more than twenty-five years at the elementary school level. He is a successful author and a member of the Authors Guild and the National Writers Association. In addition to math resources, he has written several resources for English and writing teachers, among them *Writing Workshop Survival Kit* (1993; second edition, 2005); *The Writing Teacher's Book of Lists* (1991; second edition, 2004); *Ready-to Use Reading Proficiency Lessons and Activities, 10th Grade Level* (2003); *Ready-to-Use Reading Proficiency Lessons and Activities, 8th Grade Level* (2002); *Ready-to-Use Reading Proficiency Lessons and Activities, 4th Grade Level* (2002); *Reading Workshop Survival Kit* (1997); and *English Teacher's Great Books Activities Kit* (1994), all published by Jossey-Bass.

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JOSSEY-BASS TEACHER

Jossey-Bass Teacher provides educators with practical knowledge and tools to create a positive and lifelong impact on student learning. We offer classroom-tested and research-based teaching resources for a variety of grade levels and subject areas. Whether you are an aspiring, new, or veteran teacher, we want to help you make every teaching day your best.

From ready-to-use classroom activities to the latest teaching framework, our value-packed books provide insightful, practical, and comprehensive materials on the topics that matter most to K–12 teachers. We hope to become your trusted source for the best ideas from the most experienced and respected experts in the field.

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SECTION 1

Integers, Variables, and Expressions

Teaching Notes 1.1: Using the Order of Operations

The order of operations is a set of rules for simplifying expressions that have two or more operations. A common mistake students make is to perform all operations in order from left to right, regardless of the proper order.

1. Present this problem to your students: $10 - 3 \times 2 + 2 \div 2$. Ask your students to solve. Some will apply the correct order of operations and find that the answer is 5, which is correct. Others will solve the problem in order from left to right and arrive at the answer of 8. Explain that this is the reason we use the order of operations. It provides rules to follow for solving problems.
2. Explain that to simplify an expression, the order of operations must be followed. State the following rules for the order of operations:

   - Perform all multiplication and division in order from left to right.
   - Perform all addition and subtraction in order from left to right.

   Emphasize that multiplication and division must be done first, no matter where these symbols appear in the expression.
3. Provide some examples, such as those below. Ask your students what steps they would follow to simplify the expressions. Then ask them to simplify each example.

$3 + 2 \times 5 \div 5$	Steps: $\times, \div, +$	Answer is 5.
$15 \div 5 \div 3 \times 2$	Steps: $\div, \div, \times$	Answer is 2.
$12 - 2 \div 2 \times 8$	Steps: $\div, \times, -$	Answer is 4.

4. Review the information and examples on the worksheet with your students.

EXTRA HELP:

Be sure you have performed all of the operations in their proper order.

ANSWER KEY:

(1) 5 **(2)** 21 **(3)** 4 **(4)** 18 **(5)** 24 **(6)** 188 **(7)** 1 **(8)** 39 **(9)** 8 **(10)** 48 **(11)** 10 **(12)** 16

(Challenge) The subtraction symbol should be inserted in the blank.

Name _____ Date _____

WORKSHEET 1.1: USING THE ORDER OF OPERATIONS

Mathematicians have agreed to simplify expressions that have no exponents or grouping symbols according to the following rules:

1. Multiply and divide in order from left to right.
2. Start at the left again and add and subtract from left to right.

EXAMPLES

$50 - 3 \times 2 \times 5 =$	$10 + 2 - 3 \times 4 =$	$32 \div 2 - 2 \times 3 =$
$50 - 6 \times 5 =$	$10 + 2 - 12 =$	$16 - 2 \times 3 =$
$50 - 30 =$	$12 - 12 =$	$16 - 6 =$
20	0	10

DIRECTIONS: Simplify each expression.

1. $12 - 2 \times 4 + 1$

2. $12 \times 4 \div 2 - 3$

3. $10 \times 2 - 2 \times 8$

4. $10 \times 2 - 6 \div 3$

5. $8 + 1 + 6 \times 5 \div 2$

6. $48 \div 2 \times 8 - 4$

7. $15 - 2 - 2 \times 6$

8. $35 + 8 - 12 \div 3$

9. $3 \times 7 - 8 - 5$

10. $20 \times 2 + 10 - 4 \div 2$

11. $8 - 4 + 2 \times 3$

12. $40 \div 8 - 5 + 3 \times 2 + 10$

CHALLENGE: Place the correct operation symbol in the blank so that $12 ___ 3 \times 2 + 2 = 8$.

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Teaching Notes 1.2: Simplifying Expressions That Have Grouping Symbols

If an expression contains grouping symbols, the order of operations requires that whatever part of the expression is contained in the grouping symbols be simplified first. A common mistake of students is to ignore the grouping symbols when simplifying.

1. Explain that grouping symbols are sometimes used to enclose an expression. There are several types of grouping symbols, including parentheses, brackets, and the fraction bar. Parentheses are the most common.
2. Explain the meaning of grouping symbols. For example, $3 \times (4 + 2)$ means 3 groups of 6 which equals 18. Emphasize that this is quite different from $3 \times 4 + 2$, which means 3 groups of 4 plus 2 more and is equal to 14. Have your students solve each problem. Discuss why each provides a different answer.
3. Explain that all operations within parentheses should be done first, following the order of operations.
4. Review the steps for the order of operations and the examples on the worksheet with your students.

EXTRA HELP:

The multiplication sign is often omitted before a grouping symbol. Example: $3(5 + 4)$ is the same as $3 \times (5 + 4)$.

ANSWER KEY:

(1) 36 **(2)** 2 **(3)** 52 **(4)** 3 **(5)** 104 **(6)** 4 **(7)** 0 **(8)** 22

(Challenge) Yes. The parentheses are not necessary. Each expression equals 3.

Name _____ Date _____

WORKSHEET 1.2: SIMPLIFYING EXPRESSIONS THAT HAVE GROUPING SYMBOLS

Common grouping symbols include parentheses ( ), brackets [ ], and the fraction bar —. Follow the steps below to simplify expressions with grouping symbols:

1. Simplify expressions within grouping symbols first by following the order of operations. Multiply and divide in order from left to right. Then add and subtract in order from left to right.
2. After you have simplified all expressions within grouping symbols, multiply and divide in order from left to right.
3. Add and subtract in order from left to right.

EXAMPLES

$14 - (8 - 6) \times 5 =$	$2[3 + 5 \times 4 - 1] =$	$1 + \frac{3 + 12}{3} =$
$14 - 2 \times 5 =$	$2[3 + 20 - 1] =$	$1 + \frac{15}{3} =$
$14 - 10 =$	$2[23 - 1] =$	$1 + 5 =$
4	$2[22] = 44$	6

DIRECTIONS: Simplify the following expressions.

1. $(8 + 4) \times 3$
2. $28 \div (7 \times 2)$
3. $80 - 2[6 + 4 \times 2]$
4. $(3 + 6) - (2 + 4)$
5. $12 + 4(8 + 3 \times 5)$
6. $\frac{15 - 1}{7} \times 2$
7. $3(4 + 6) - 5(10 - 4)$
8. $15 + \frac{2 + 6 \times 2}{2}$

CHALLENGE: Is $3 \times 2 - 12 \div 4$ the same as $(3 \times 2) - (12 \div 4)$? Explain your reasoning.

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Teaching Notes 1.3: Simplifying Expressions with Nested Grouping Symbols

If an expression has nested grouping symbols — one or more sets of grouping symbols inside another — some students ignore the innermost symbols. They then go on to simplify the expression incorrectly.

1. Explain that grouping symbols help to indicate what operations to do first when solving expressions.
2. Explain that when a grouping symbol is set within another, the expression within the innermost grouping symbol must be simplified first. Provide the following example: $12 - (5 + (3 \times 2))$. Explain that there are two sets of parentheses in this problem. Operations in the inner set of parentheses must be completed first and then work is completed outward. Demonstrate this by first solving 3×2 and replacing the answer, 6, in the new problem: $12 - (5 + 6)$. The correct answer is 1.
3. Emphasize that students should always work outward from the nested grouping symbol, following the order of operations. Depending on your students, you may want to review the order of operations:
   - Multiply and divide from left to right.
   - Add and subtract from left to right.
4. Review the steps for simplifying and the examples on the worksheet with your students. Note the use of grouping symbols and particularly the innermost grouping symbols.

EXTRA HELP:

Parentheses, brackets, and fraction bars are examples of grouping symbols.

ANSWER KEY:

(1) 24 **(2)** 22 **(3)** 120 **(4)** 80 **(5)** 91 **(6)** 220 **(7)** 4 **(8)** 3

(Challenge) 4

Name _____ Date _____

WORKSHEET 1.3: SIMPLIFYING EXPRESSIONS WITH NESTED GROUPING SYMBOLS

Sometimes an expression has one or more grouping symbols inside another. These are often called "nested" grouping symbols. Follow the steps below when simplifying expressions with nested grouping symbols:

1. Simplify the expressions within the nested grouping symbols first.
2. After simplifying the innermost expression, work outward.
3. Simplify the expression according to the order of operations. Multiply and divide from left to right. Then add and subtract from left to right.

EXAMPLES

$20 - [3 \times (14 - 12)] =$	$4[(6 + 3) \times 10] =$	$\frac{3 + 24}{12 - (10 - 7)} =$
$20 - [3 \times 2] =$	$4(9 \times 10) =$	$\frac{3 + 24}{12 - 3} =$
$20 - 6 =$	$4(90) =$	$\frac{27}{9} =$
14	360	3

DIRECTIONS: Simplify.

1. $4[9 - (5 - 2)]$
2. $2[4 + 7(4 - 3)]$
3. $2[3(12 - 7) \times 4]$
4. $4[8(6 - 3) - 4]$
5. $3 + 2[4(3 + 8)]$
6. $(3 + 2)[4(3 + 8)]$
7. $\frac{2(3 + 7)}{3 + 2}$
8. $\frac{6 \times 8}{2(3 + 5)}$

CHALLENGE: What is the missing number? $6 + [___(3 + 3(8 + 1))] = 126$

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Teaching Notes 1.4: Using Positive Exponents and Bases Correctly

Many students make mistakes when working with positive exponents and bases. One of the most common errors is equating x^n with $x \times n$.

SPECIAL MATERIALS

Graph paper

1. Explain that an exponent represents the number of times a base is used as a factor. For example, $5^2 = 5 \times 5$ and $5^3 = 5 \times 5 \times 5$. Emphasize that 5^2 does not equal 5×2 or 2×5 and 5^3 does not equal 5×3 or 3×5.
2. Ask your students to draw a square, five units per side, on graph paper.
3. Instruct them to count the number of small squares inside the large square.
4. Explain that they should count twenty-five small squares. These squares represent 5×5 or 5^2. Emphasize that 5 is a factor two times, which is the meaning of 5^2, pronounced "five squared." It is termed "squared" because when modeled geometrically 5^2 forms a square. This may help your students remember that it is 5 times 5, not 5 times 2. Likewise, 5 to the third power is often called "five cubed." When modeled geometrically, 5^3 forms a cube with five units on each edge.
5. Next ask your students to draw a rectangle, five units long and two units wide, on graph paper. They should count ten small squares inside the rectangle. These squares represent 5×2, which is quite different from 5^2.
6. Review the examples on the worksheet with your students. Emphasize that in the first example 4 is a factor 3 times. In the second example 3 is a factor 5 times.

EXTRA HELP:

x^n means x is a factor n times.

ANSWER KEY:

(1) 16, 8 **(2)** 18, 81 **(3)** 81, 12 **(4)** 12, 64 **(5)** 21, 343 **(6)** 100, 20 **(7)** 10, 32
(8) 1, 3 **(9)** 25, 10 **(10)** 18, 216

(Challenge) 2, because $2^2 = 4$ and $2 \times 2 = 4$.

Name _____ Date _____

WORKSHEET 1.4: USING POSITIVE EXPONENTS AND BASES CORRECTLY

An exponent indicates the number of times its base is used as a factor. In 3^2, 3 is the base and 2 is the exponent.

EXAMPLES

$4^3 = 4 \times 4 \times 4 = 64$ $3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$

DIRECTIONS: Find the value of each expression.

1. 2^4 4×2

2. 2×9 9^2

3. 3^4 4×3

4. 2×6 2^6

5. 3×7 7^3

6. 10^2 10×2

7. 5×2 2^5

8. 1^3 3×1

9. 5^2 5×2

10. 6×3 6^3

CHALLENGE: $x^2 = 2x$ is true for only one positive integer. What number makes this equation true? Explain how you know your answer is correct.

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Teaching Notes 1.5: Simplifying Expressions with Grouping Symbols and Exponents

Expressions that involve exponents, parentheses, or several operations are often confusing to students. To ensure that your students become proficient in simplifying such expressions, reinforcement of the order of operations is essential.

1. Explain that some expressions contain exponents. Depending on the abilities of your students, you might find it helpful to review 1.4: "Using Positive Exponents and Bases Correctly."
2. Explain to your students that the order of operations may become confusing when they must compute using multiple operations. Suggest that students use the acronym "Please excuse my dear Aunt Sally" to help them remember the order of operations for expressions with grouping symbols and exponents:
   - P stands for parentheses (or grouping symbols).
   - E stands for exponents.
   - M stands for multiplication.
   - D stands for division.
   - A stands for addition.
   - S stands for subtraction.

   Note that although multiplication precedes division in the acronym, these operations must be completed in order from left to right. Therefore, there will be times students will divide before multiplying. Similarly, addition precedes subtraction in the acronym, and these operations must also be completed in order from left to right. There will be times students will subtract before adding.
3. Review the steps for using the order of operations and the examples on the worksheet with your students.

EXTRA HELP:

Suggest that students rewrite each problem after they have completed an operation. This will help them organize their work and avoid mistakes.

ANSWER KEY:

(1) 13 **(2)** 112 **(3)** 43 **(4)** 19 **(5)** 46 **(6)** 26 **(7)** 3 **(8)** 3

(Challenge) 7

Name _____ Date _____

WORKSHEET 1.5: SIMPLIFYING EXPRESSIONS WITH GROUPING SYMBOLS AND EXPONENTS

To simplify expressions with grouping symbols, exponents, and other operations, follow the steps below:

1. Simplify expressions within grouping symbols first. Simplify the innermost expressions first and continue working outward to the outermost expressions. As you do, be sure to follow steps 2, 3, and 4.
2. Simplify powers.
3. Multiply and divide in order from left to right.
4. Add and subtract in order from left to right.

EXAMPLES

$3 + 4 \times 2 + 5^2 =$	$4^2 + (3 + 5) \times 2 =$	$3^2[(3 + 4) \times 2] =$
$3 + 4 \times 2 + 25 =$	$4^2 + 8 \times 2 =$	$3^2[(7 \times 2)] =$
$3 + 8 + 25 =$	$16 + 8 \times 2 =$	$3^2 \times 14 =$
$11 + 25 =$	$16 + 16 =$	$9 \times 14 =$
36	32	126

DIRECTIONS: Simplify.

1. $2 \times 3^2 - 5$

2. $37 + 5^2 \times 3$

3. $2^3 + 7(8 - 3)$

4. $(2 + 1)^3 - 2^3$

5. $(13 - 2^3) \times 2 + 6^2$

6. $2^2 \times 8 - 5 - 1$

7. $\dfrac{48}{2^4}$

8. $\dfrac{(5 - 2)^3}{2 \times 7 - 5}$

CHALLENGE: What is the missing number? $\dfrac{4(____ + 3)^2}{5(14 - 3^2)} = 16$

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Teaching Notes 1.6: Evaluating Expressions

Evaluating an expression requires students to replace each variable in an expression with a given value and simplify the result. Common errors occur when students either substitute an incorrect value for the variable or follow the order of operations incorrectly.

1. Review variables by explaining that a variable represents an unknown quantity. It is usually expressed as a letter.
2. Explain that sometimes students are required to find a variable's value. At other times the value of a variable is provided. When the value of a variable is provided, students must replace the variable in the expression with that value.
3. Stress to your students the importance of substituting values for variables correctly.
4. Encourage them to rewrite the problem after they have substituted the correct values.
5. Review the order of operations and examples on the worksheet with your students. Caution them to pay close attention to nested grouping symbols. Depending on their abilities, you may find it helpful to review 1.3: "Simplifying Expressions with Nested Grouping Symbols."

EXTRA HELP:

A number directly before a variable denotes multiplication. For example, $3a$ means 3 times a.

A number or variable above or below a fraction bar denotes division. For example, $\frac{a}{4}$ means a number divided by 4.

ANSWER KEY:

(1) 7 **(2)** 58 **(3)** 56 **(4)** 64 **(5)** 2 **(6)** 29 **(7)** 14 **(8)** 26

(Challenge) Answers may vary. One acceptable response is $c(d - a) - b$.

Name _____ Date _____

WORKSHEET 1.6: EVALUATING EXPRESSIONS

To evaluate an expression means to replace a variable or variables with a given number or numbers and then simplify the expression. Follow the steps below:

1. Rewrite the expression by replacing all the variables with the given values. Be sure you have substituted correctly.
2. Follow the order of operations for simplifying:
   - Simplify expressions within grouping symbols first. If there are nested grouping symbols, simplify the innermost first, then work outward.
   - Simplify powers.
   - Multiply and divide in order from left to right.
   - Add and subtract in order from left to right.

EXAMPLES

$a = 3, b = 4, c = 5$, and $d = 6$.

$2a + 10c =$	$2a^2 + cd =$	$2(b + c^2) =$
$2 \times 3 + 10 \times 5 =$	$2 \times 3^2 + 5 \times 6 =$	$2(4 + 5^2) =$
$6 + 50 =$	$2 \times 9 + 5 \times 6 =$	$2(4 + 25) =$
56	$18 + 30 =$	$2(29) =$
	48	58

DIRECTIONS: Evaluate each expression if $a = 3, b = 4, c = 5$, and $d = 6$.

1. $ab - c$
2. $8d + 2c$
3. $8(a + b)$
4. $(a + c)^2$
5. $\frac{ab}{d} =$
6. $c(a + b) - d$
7. $d + 2[a + (c - b)]$
8. $\frac{cd}{a} + b^2$

CHALLENGE: Use the values and variables above to create an expression that equals 11.

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Teaching Notes 1.7: Writing Expressions

Writing expressions is a prerequisite skill to writing equations. Most of the errors students make in writing expressions arise from misinterpreting words and phrases, particularly those having to do with subtraction and division.

1. Explain that key words often signal addition, subtraction, multiplication, and division. Following are some examples:
   - *Addition:* add, total, in all, combine, sum, increased by
   - *Subtraction:* less than, more than, subtract, difference, decreased by
   - *Multiplication:* product, multiply, of, twice, double, triple
   - *Division:* divide, quotient, split, groups of, quarter
2. Direct your students to focus their attention on subtraction and division. Point out that unlike addition and multiplication, subtraction and division are not commutative; the proper order of the terms cannot be switched.
3. Provide the following example: 4 less than a number n. Ask your students to write an expression for this phrase, then discuss the answer. Explain that although 4 comes first in the phrase, it must be placed after the n in the expression. The correct expression for the phrase 4 less than n is $n - 4$. It cannot be $4 - n$. Offer this illustration: $6 - 4 \neq 4 - 6$.
4. Provide this example: A number n divided by 2. Ask your students to write an expression for this phrase. It is $n \div 2$. Note that it cannot be $2 \div n$. Offer this illustration: $4 \div 2 \neq 2 \div 4$.
5. Review the chart on the worksheet with your students. You might ask your students to generate more examples.

EXTRA HELP:

To check your work when writing expressions, pick a number, substitute it for the variable, and see if the result is reasonable.

ANSWER KEY:

(1) $n + 6$ or $6 + n$ **(2)** $n - 1$ **(3)** $3n$ **(4)** $n - 8$ **(5)** $n \div 10$ **(6)** $n - 9$ **(7)** $n - 6$ **(8)** $3 + n$ or $n + 3$

(Challenge) Answers may vary. An acceptable response is 3 less than twice a number.

Name _____ Date _____

WORKSHEET 1.7: WRITING EXPRESSIONS

The chart below illustrates some expressions and examples of their meanings. n represents a number.

ADDITION: $n + 3$	SUBTRACTION: $n - 3$
a number increased by 3	a number decreased by 3
3 more than a number	3 less than a number
a number plus 3	a number minus 3
the sum of a number and 3	a number reduced by 3
MULTIPLICATION: $2n$	**DIVISION: $n \div 2$**
the product of 2 and a number	a number divided by 2
2 times a number	half of a number
2 multiplied by a number	the quotient when a number is divided by 2

DIRECTIONS: Write an expression for each phrase. Use n to represent the number.

1. 6 greater than a number

2. A number minus 1

3. 3 times a number

4. 8 less than a number

5. A number divided by 10

6. A number decreased by 9

7. A number minus 6

8. The sum of a number and 3

CHALLENGE: Write a phrase for this expression: $2n - 3$.

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Teaching Notes 1.8: Writing Expressions Involving Grouping Symbols

Some expressions describe operations as a sum, difference, product, or quotient. To write expressions like these, students may have to include grouping symbols. Ignoring necessary grouping symbols is a common error.

1. Discuss basic examples of expressions:
   - 3 times a number plus 2 can be written as $3n + 2$.
   - 3 times the sum of a number and 2 can be written as $3(n + 2)$.
2. Explain that in the first example the number is multiplied by 3, then 2 is added. In the second example, the sum of the number and 2 is multiplied by 3. $n + 2$ must be written in parentheses.
3. Emphasize that these two expressions have different values. For example, if $n = 4$, $3n + 2 = 14$ and $3(n + 2) = 18$.
4. Encourage your students to consider whether an expression refers to a quantity or only one number. Remind them that the words "sum," "difference," "product," and "quotient" often signify that grouping symbols are needed.
5. Review the examples on the worksheet with your students.

EXTRA HELP:

Quantities must be written in grouping symbols.

ANSWER KEY:

(1) $\frac{n+12}{5}$ **(2)** $n + \frac{12}{5}$ **(3)** $(4n)^3$ **(4)** $4n^3$ **(5)** $2(n - 10)$ **(6)** $2n - 10$ **(7)** $(4 + n)^2$ **(8)** $4 + n^2$

(Challenge) $\frac{x+y}{2}$

Name _____ Date _____

WORKSHEET 1.8: WRITING EXPRESSIONS INVOLVING GROUPING SYMBOLS

An expression for a quantity that is added, subtracted, multiplied, or divided must be written within a grouping symbol. Key words such as "sum," "difference," "product," and "quotient" often indicate two or more numbers that are usually written within grouping symbols.

EXAMPLES

3 times a number squared: $3n^2$. Only the number, n, is squared.
The product of 3 times a number, squared: $(3n)^2$. The quantity, $3n$, is squared.

DIRECTIONS: Write an expression for each phrase. Use n to represent a number.

1. The sum of a number and 12, divided by 5

2. A number plus 12 divided by 5

3. The product of 4 times a number, cubed

4. 4 times a number cubed

5. Twice the difference when 10 is subtracted from a number

6. Two times a number minus 10

7. The sum of 4 and a number, squared

8. 4 plus a number squared

CHALLENGE: Write an expression to show the average of x and y.

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Teaching Notes 1.9: Identifying Patterns by Considering All of the Numbers

Identifying patterns and recognizing the relationship between numbers is an important skill. Students frequently make a quick decision based on examining only a few numbers in a pattern instead of all of them, which, of course, leads to mistakes.

1. Explain that the numbers in any pattern are related in some way.
2. Emphasize that students must consider all, not merely some, of the numbers of a pattern before they can identify the relationship between the numbers in the pattern.
3. Provide your students with the following example: 2, 4, 8, . . .
4. Ask them to consider only the first two numbers. Explain that two relationships for the numbers are possible:
   - Each number is two more than the previous one.
   - Each number is two times the previous one.
5. Direct your students to now consider the third number. Explain that the third number proves that the only correct relationship is that each number is two times the number before it.
6. Review the information and examples on the worksheet with your students.

EXTRA HELP:

All numbers in a pattern must be related.

ANSWER KEY:

(1) 3 is subtracted from the previous number.
(2) 8 is added to the previous number.
(3) The second number is found by adding 10, the third is found by adding 1, and this pattern continues.
(4) Numbers are found by adding 3, then adding 4, adding 5, and so on.
(5) Each number is the square of the previous number.
(6) Each number is one-third of the previous number.
(7) Each number is 3 more than the previous number.
(8) Each number is found by subtracting 1, then adding 1, and so on.
(9) Each number is a multiple of 4.
(10) Each number is found by dividing the previous number by 5.

(Challenge) The missing number is 11. Each number is the sum of the two preceding numbers.

Name _____ Date _____

WORKSHEET 1.9: IDENTIFYING PATTERNS BY CONSIDERING ALL OF THE NUMBERS

A pattern is a group of numbers that are related in some way. Following are some hints for recognizing patterns:

- If the numbers are increasing, consider the operations of addition and multiplication.
- If the numbers are decreasing, consider the operations of subtraction and division.
- Some number patterns involve a combination of operations.

EXAMPLES

4, 6, 8, 10, . . . Each number is 2 more than the preceding number.
100, 50, 25, 12.5, . . . Each number is half of the preceding number.
10, 19, 37, 73, . . . Each number is found by multiplying the previous number by 2 and subtracting 1.

DIRECTIONS: Identify the pattern in each sequence below.

1. 64, 61, 58, 55, . . .

2. 2, 10, 18, 26, . . .

3. 2, 12, 13, 23, 24, . . .

4. 3, 6, 10, 15, 21, . . .

5. 5, 25, 625, . . .

6. 729, 243, 81, 27, . . .

7. 0, 3, 6, 9, . . .

8. 19, 18, 19, 18, . . .

9. 4, 8, 12, 16, . . .

10. 1,000, 200, 40, . . .

CHALLENGE: What is the missing number in this pattern?
3, 4, 7, ___, 18, 29, 47, . . .

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Teaching Notes 1.10: Writing Prime Factorization

Expressing a composite number as a product of its prime factors is called writing the prime factorization. A common error students make is failing to complete the prime factorization of a number by mistaking a composite number for a prime.

1. Make sure your students understand the definition of a prime number. A prime number is a whole number greater than 1 that has only two factors: 1 and itself. Compare this definition with that of a composite number: a whole number greater than 1 that has more than two factors. Note that 1 is neither prime nor composite.
2. Point out the first ten prime numbers on the worksheet. Ask your students to volunteer examples of composite as well as other prime numbers.
3. Explain that every composite number can be uniquely expressed as the product of prime factors. The order of the prime factors does not matter. $15 = 3 \times 5$ or 5×3.
4. Review the information and example on the worksheet with your students. Note that all of the factors in prime factorization are prime numbers and that the factors are written in ascending order. Also note how the prime factors can be written with exponents. For example, the prime factorization of 20 can be written as $2 \times 2 \times 5$ or $2^2 \times 5$.

EXTRA HELP:

2 is a prime factor of every even number.

ANSWER KEY:

(1) $2 \times 5 \times 7$ **(2)** $2^2 \times 7$ **(3)** $2^2 \times 3 \times 5$ **(4)** $2^2 \times 5^2$ **(5)** 3^4 **(6)** $3^2 \times 5$
(7) 3×5^2 **(8)** $2^3 \times 3$ **(9)** $2^2 \times 3 \times 5^2$ **(10)** $2 \times 3^2 \times 7$

(Challenge) $6 = 2 \times 3$. 2 and 3 are the two smallest prime numbers.

Name _____ Date _____

WORKSHEET 1.10: WRITING PRIME FACTORIZATION

Every composite number can be written as a product of prime numbers. This is called "prime factorization." To write the prime factorization of a composite number, follow these steps:

1. Find two factors of the composite number.
2. Factor each of the factors (if possible).
3. Continue this process until all of the factors are prime numbers.

Here are the first ten prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29.

EXAMPLE

Find the prime factorization of 200. (*Note:* In the example, the prime numbers that are used to write the prime factorization are underlined.)

- Two factors of 200 are 10 and 20. $10 \times 20 = 200$.
- The prime factors of 10 are 2 and 5. $\underline{2} \times \underline{5} = 10$.
- Two factors of 20 are 4 and 5. $4 \times \underline{5} = 20$. 5 is a prime number but 4 is a composite number and must be factored.
- The two prime factors of 4 are 2 and 2. $\underline{2} \times \underline{2} = 4$.
- Write the prime factorization by writing all of the prime factors. Therefore, the prime factorization of $200 = 2 \times 2 \times 2 \times 5 \times 5$ or $2^3 \times 5^2$.

DIRECTIONS: Find the prime factorization of each number.

1. 70 **2.** 28 **3.** 60 **4.** 100 **5.** 81

6. 45 **7.** 75 **8.** 24 **9.** 300 **10.** 126

CHALLENGE: What is the smallest number that can be expressed as the product of two different prime numbers?

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Teaching Notes 1.11: Finding the Greatest Common Factor

When students are asked to find the greatest common factor (GCF) of two numbers via the method of prime factorization, confusion may arise over the use of the smallest exponent of a common base or bases. Because they are finding the "greatest" common factor, some students mistakenly believe they must use the greatest exponent of a common base or bases.

1. Explain that the greatest common factor is the largest common factor of two or more numbers.
2. Instruct your students to find the greatest common factor of 315 and 135. Work through the process together.
3. Ask for volunteers to provide the prime factorization of 315 and 135:
   - $315 = 3 \times 3 \times 5 \times 7$
   - $135 = 3 \times 3 \times 3 \times 5$
4. Ask your students to list the factors that are common to 315 and 135, and then find the product. $3 \times 3 \times 5 = 45$. The GCF of 315 and 135 is 45.
5. Explain that prime factorization with exponents can be used to find the GCF. Note that 315 and 135 have a common factor of 3^2. They also have a common factor of 5. Therefore, $3^2 \times 5 = 45$, which is the GCF.
6. Review the information and example on the worksheet with your students and emphasize why $2^2 \times 3$ is the GCF.

EXTRA HELP:

If there are no common factors listed in the prime factorization of two numbers, the GCF is 1. Although 1 is not a prime number, it is a factor of every number.

ANSWER KEY:

(1) 3 **(2)** 5 **(3)** 6 **(4)** 12 **(5)** 6 **(6)** 10 **(7)** 5 **(8)** 20 **(9)** 5 **(10)** 1 **(11)** 2 **(12)** 9

(Challenge) Disagree. Explanations may vary. One explanation is that 1 is the GCF of any two prime numbers. But 1 may also be the GCF of two numbers that are composite. For example, 1 is the GCF of 4 and 15. Also, 1 may be the GCF of a prime and composite number. For example, 1 is the GCF of 3 and 10.

Name _____ Date _____

WORKSHEET 1.11: FINDING THE GREATEST COMMON FACTOR

The greatest common factor (GCF) is the greatest factor that two (or more) numbers have in common. To find the GCF of two numbers, follow the steps below:

1. Use exponents to write the prime factorization of each number.
2. Find the prime numbers that are factors of both numbers.
3. Choose the smaller exponent of each common prime factor. The base with the smaller exponent is always a factor of the same base with a larger exponent.
4. The product of the common factors is the GCF.

EXAMPLE

Find the GCF of 72 and 60.

$72 = 2^3 \times 3^2$ $\quad$ $60 = 2^2 \times 3 \times 5$

Common factors are 2 and 3.
2 is the smaller exponent of 2. (2^2 is a factor of 2^3.)
1 is the smaller exponent of 3. (3^1 is a factor of 3^2.)
The GCF of 72 and 60 $= 2^2 \times 3 = 12$.

DIRECTIONS: Find the GCF of each pair of numbers.

1. 9 and 12
2. 25 and 35
3. 6 and 60
4. 36 and 192
5. 36 and 30
6. 90 and 400
7. 135 and 50
8. 100 and 140
9. 45 and 200
10. 67 and 9
11. 48 and 50
12. 9 and 72

CHALLENGE: Do you agree with the following statement? If the GCF of two numbers is 1, then both numbers must be prime. Explain your answer.

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Teaching Notes 1.12: Finding the Least Common Multiple

Students often confuse the least common multiple (LCM) with the greatest common factor (GCF). Reinforcing the meaning of factors, multiples, and common multiples can reduce confusion.

1. Discuss the meaning of the greatest common factor (GCF) of two numbers: the largest factor that two numbers have in common. For example, the GCF of 8 and 12 is 4.
2. Illustrate the concepts of multiples and the least common multiple by showing multiples of 30 and 12:
   - Ask your students to list multiples of 30: 30, 60, 90, 120, 150, 180, 210, . . .
   - Ask them to list multiples of 12: 12, 24, 36, 48, 60, 72, 84, 96, . . .
3. Ask your students to identify the first number that is common to both lists, which is 60. 60 is the least common multiple of 30 and 12. (You might mention that if the lists were extended more common multiples would be found.) Note that this is a somewhat tedious method, especially for large numbers.
4. Explain to your students that rather than list multiples to find the LCM of two numbers, they can use prime factorization. Instruct them to write the prime factorization of 30 and 12. $30 = 2 \times 3 \times 5$ and $12 = 2^2 \times 3$. They can find the LCM by finding the product of each base raised to the highest power of each prime factor. The LCM of 30 and 12 is $2^2 \times 3 \times 5 = 60$.
5. Review the information and example on the worksheet with your students.

EXTRA HELP:

The LCM of two numbers will always be greater than or equal to the larger number.

ANSWER KEY:

(1) 60 **(2)** 120 **(3)** 80 **(4)** 720 **(5)** 108 **(6)** 90 **(7)** 75 **(8)** 448 **(9)** 345 **(10)** 240 **(11)** 24 **(12)** 1,430

(Challenge) The statement is true. Explanations may vary. An acceptable response is if 1 is the only common factor the LCM is found by multiplying all of the numbers in the prime factorization. *Example:* 15 and 14 have no common factors other than 1. $15 = 3 \times 5$ and $14 = 2 \times 7$. Therefore the LCM of 15 and 14 is $2 \times 3 \times 5 \times 7 = 210$.

Name _____ Date _____

WORKSHEET 1.12: FINDING THE LEAST COMMON MULTIPLE

The least common multiple (LCM) is the smallest number that is a multiple of two or more numbers. To find the LCM of two numbers, do the following:

1. Write the prime factorization of each number.
2. Find the product of each prime number raised to the largest power.

EXAMPLE

Find the LCM of 24 and 180.

$24 = 2^3 \times 3$ $\qquad$ $180 = 2^2 \times 3^2 \times 5$

In the prime factorization of 24 and 180, the largest power of 2 is 2^3 and the largest power of 3 is 3^2. The only power of 5 is 5^1, which is 5. The LCM of 24 and 180 is $2^3 \times 3^2 \times 5 = 360$.

DIRECTIONS: Find the LCM of each pair of numbers.

1. 15, 12 **2.** 20, 24 **3.** 16, 10 **4.** 45, 48

5. 27, 36 **6.** 18, 90 **7.** 25, 75 **8.** 64, 28

9. 115, 15 **10.** 6, 80 **11.** 4, 24 **12.** 11, 130

CHALLENGE: Do you agree with the following statement? If two numbers have no common factor other than 1, then their product is the LCM. Explain your answer.

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Teaching Notes 1.13: Classifying Counting Numbers, Whole Numbers, and Integers

Although there are several types of numbers, students often have trouble classifying three of the most common: counting numbers, whole numbers, and integers. Many students simply lump these three types of numbers together.

1. Ask for a volunteer to give an example of the counting numbers. Most likely the student will say 1, 2, 3, 4, 5, . . . Write the list on the board and explain that the ellipsis means that the counting numbers continue in this manner, adding 1 to the previous number. They go on infinitely. Offer this suggestion to your students to help them remember that the counting numbers begin with 1: When you start counting something, you start with 1.
2. Explain that zero is not a part of the counting numbers. Zero and the counting numbers make up the whole numbers. 0, 1, 2, 3, 4, 5, . . . Mention to your students that "whole" has an "0," which can be thought to represent "zero."
3. Explain that zero is necessary to some applications of math, as are negative numbers. For example, on the Celsius scale, a subfreezing temperature on a cold winter day might be -5° or five degrees less than zero. The counting numbers, zero, and the opposites of the counting numbers are integers. . . . $-3, -2, -1, 0, 1, 2, 3, \ldots$
4. Review the examples of counting numbers, whole numbers, and integers on the worksheet with your students. Note that the counting numbers are a subset of the whole numbers and the whole numbers are a subset of the integers.

EXTRA HELP:

Think of whole numbers as the counting numbers and zero.

ANSWER KEY:

(Counting numbers) 15, 1, 4, 12, 3, 23, 154, 752, 19 (Whole numbers) 15, 1, 0, 4, 12, 3, 23, 154, 752, 19
(Integers) 15, −8, 1, 0, −10, −7, 4, 12, 3, −1, 23, 154, −30, 752, 19

(Challenge) Answers may vary. Possible answers include below zero temperatures, a loss of yardage in a football game, a decrease in business activity, or the deduction of points on a test.

Name _____ Date _____

WORKSHEET 1.13: CLASSIFYING COUNTING NUMBERS, WHOLE NUMBERS, AND INTEGERS

Numbers can be classified into various groups. Three of the most important groups are shown below:

- *Counting numbers:* These are the numbers we count with. Starting with 1, they continue forever by adding 1 to the previous number: 1, 2, 3, 4, 5, . . .
- *Whole numbers:* Zero and the counting numbers make up the whole numbers: 0, 1, 2, 3, 4, 5, . . .
- *Integers:* Negative and positive whole numbers and zero make up the integers: . . . −3, −2, −1, 0, 1, 2, 3, . . .

Most numbers belong to more than one of the groups above. For example, 3 is a counting number, a whole number, and an integer. But −3 is only an integer.

DIRECTIONS: Classify the numbers below as counting numbers, whole numbers, and integers. Some numbers may be members of more than one group.

15	−8	1	0	−10
−7	4	12	3	−1
23	154	−30	752	19

CHALLENGE: Give at least three examples of how negative numbers may be used.

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Teaching Notes 1.14: Finding Absolute Values and Opposites

Two terms associated with integers are "absolute value" and "opposite." These terms are easily confused. Sometimes the opposite of an integer and its absolute value are the same; sometimes they are different.

1. Explain that the absolute value of a number is the number of units the number is from 0 on the number line. Offer this example: the absolute value of both −2 and 2 is 2, because both numbers are 2 units from 0.
2. Model absolute value on the number line. Show that −2 is two spaces from 0. Therefore, its absolute value is 2. Show that 3 is three spaces from 0. Therefore, its absolute value is 3. Remind students that the absolute value of a number is always positive. This is because it represents the distance from 0. Distances cannot be negative.
3. Explain that the opposite of a number always has the opposite sign, except for 0, which is neither positive nor negative. The opposite of −8 is 8 and opposite of 3 is −3. Note that the sum of opposite integers is always 0.
4. Review the number line and information on the worksheet with your students. Point out the following:
   - Every integer is the same distance from the number to its right or left.
   - Positive numbers are to the right of zero.
   - Negative numbers are to the left of zero.

EXTRA HELP:

Negative numbers may be written with a lowered minus sign, −7, or a raised minus sign, ⁻7.

ANSWER KEY:

(1) −6 **(2)** 4 **(3)** 1 **(4)** 5 **(5)** 10 **(6)** 12 **(7)** 20 **(8)** 8 and −8

(Challenge) Explanations may vary. Absolute value is the number of units a number is from zero on the number line. Opposite numbers are the same distance from zero, one to the right, the other to the left.

Name _____ Date _____

WORKSHEET 1.14: FINDING ABSOLUTE VALUES AND OPPOSITES

A number line is a way to picture numbers and points on a graph.

Each pair of numbers that are the same distance from 0 are opposites. For example, 1 is the opposite of −1 and −3 is the opposite of 3. The absolute value of a number is the number of units the number is from 0 on the number line. The direction does not matter.

EXAMPLES

$|2|$ (read "the absolute value of 2") $= 2$ because 2 is 2 units from 0.

$|-2| = 2$ because -2 is 2 units from 0.

$|0| = 0$ because 0 is 0 units from 0.

The opposite of -7 is 7; the opposite of 5 is -5.

DIRECTIONS: Find the integer or integers described below.

1. The opposite of 6

2. The absolute value of -4

3. The absolute value of 1

4. The opposite of -5

5. The opposite of -10

6. The absolute value of 12

7. The opposite of -20

8. Two numbers whose absolute value is 8

CHALLENGE: Explain how opposites and absolute values both relate to distance.

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Teaching Notes 1.15: Adding Integers with Different Signs

Adding two integers with different signs may seem like subtraction of integers to some students. They do not realize that when adding a positive integer and a negative integer, the negative integer "negates" the value of the positive integer.

1. Ask your students to imagine that a student gained 50 points while playing a video game and then lost 10 points. Explain that the loss of 10 points negates 10 of the 50-point gain. The student now has only 40 points. The 10-point loss can be represented by -10.
2. Explain that this situation can be modeled by adding integers. $50 + (-10) = 40$. The process is addition because the student gained 50 *and* lost 10. The word "and" indicates addition.
3. Now ask your students to suppose that another student lost 40 points and gained 30 in a video game. The 40-point loss has gained 30 more points, resulting in only a 10-point loss. This can also be modeled by adding integers. $-40 + 30 = -10$
4. Review the rules and examples for adding integers on the worksheet with your students.

EXTRA HELP:

Positive symbols for numbers can be omitted as a number without a sign is assumed to be positive. Negative signs must always be included.

ANSWER KEY:

(1) 17 **(2)** 9 **(3)** 0 **(4)** -8 **(5)** -7 **(6)** -24 **(7)** 62 **(8)** 49 **(9)** 5 **(10)** -25 **(11)** 14 **(12)** -12

(Challenge) Answers may vary. One acceptable response is $-10 + 3 = -7$ and $5 + (-4) = 1$.

Name _____ Date _____

WORKSHEET 1.15: ADDING INTEGERS WITH DIFFERENT SIGNS

Use the following rules to add a positive and negative integer:

1. Find the absolute value of each number.
2. Subtract the smaller absolute value from the larger.
3. Keep the sign of the integer with the greater absolute value.

EXAMPLES

$-3 + 15$	$18 + (-25)$
$\lvert -3 \rvert = 3 \quad \lvert 15 \rvert = 15$	$\lvert 18 \rvert = 18 \quad \lvert -25 \rvert = 25$
$15 - 3 = 12$	$25 - 18 = 7$
Because $\lvert 15 \rvert > \lvert -3 \rvert$, the sign is positive.	Because $\lvert -25 \rvert > \lvert 18 \rvert$, the sign is negative.
$-3 + 15 = 12$	$18 + (-25) = -7$

DIRECTIONS: Find each sum.

1. $-12 + 29$ **2.** $-1 + 10$ **3.** $8 + (-8)$

4. $-15 + 7$ **5.** $5 + (-12)$ **6.** $-38 + 14$

7. $98 + (-36)$ **8.** $-26 + 75$ **9.** $20 + (-15)$

10. $-36 + 11$ **11.** $22 + (-8)$ **12.** $-44 + 32$

CHALLENGE: Create two problems that involve adding a positive and a negative integer so that one sum is positive and the other is negative. Include the answers to your problems.

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Teaching Notes 1.16: Subtracting Integers

Subtraction of integers can be easy for students to master if they remember to rewrite the subtraction problem as an addition problem and then add the integers. Unfortunately, forgetting to rewrite the problem is a very common mistake.

1. Explain to your students that subtracting a number is defined by adding its opposite. $a - b = a + (-b)$, if a and b are integers.
2. Explain that any subtraction problem can be rewritten as an addition problem. It is often easier to add integers than subtract them.
3. Review the information and examples on the worksheet with your students. Encourage them to rewrite subtraction problems involving integers by following the steps provided. If necessary, review the steps for adding integers provided with 1.15: "Adding Integers with Different Signs."

EXTRA HELP:

The opposite of a negative number is a positive number and the opposite of a positive number is a negative number.

ANSWER KEY:

(1) -7 **(2)** -9 **(3)** 15 **(4)** 28 **(5)** 16 **(6)** -5 **(7)** -15 **(8)** 0 **(9)** -11 **(10)** 35 **(11)** -14 **(12)** -2

(Challenge) Rewrite the problem as $-3 - [-10 + (-6)]$. Find the sum of -10 and -6, which is -16. Rewrite the subtraction problem as addition: $-3 - (-16) = -3 + 16$. The sum is 13.

Name _____ Date _____

WORKSHEET 1.16: SUBTRACTING INTEGERS

You can subtract integers by rewriting the subtraction problem as addition. Follow the steps below:

1. Change the subtraction sign to addition.
2. Change the sign of the second number to its opposite.
3. Follow the rules for adding integers.

EXAMPLES

$-3 - 15 =$	$6 - 10 =$	$-5 - (-20) =$	$8 - (-2) =$
$-3 + (-15) =$	$6 + (-10) =$	$-5 + 20 =$	$8 + 2 =$
-18	-4	15	10

DIRECTIONS: Find each difference.

1. $8 - 15$

2. $3 - 12$

3. $11 - (-4)$

4. $12 - (-16)$

5. $8 - (-8)$

6. $0 - 5$

7. $-6 - 9$

8. $8 - 8$

9. $-7 - 4$

10. $20 - (-15)$

11. $-18 - (-4)$

12. $9 - 11$

CHALLENGE: Write the steps for solving the following subtraction problem. Then find the difference. $-3 - (-10 - 6)$

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Teaching Notes 1.17: Multiplying Two Integers

A common problem students have when multiplying integers is not recognizing that the product of two negative numbers is a positive number. Students may then incorrectly write a negative sign instead of the positive sign in the product.

1. Explain that the product of a positive integer and a negative integer is a negative integer. To illustrate, offer the following pattern:

   - $-1 \times 3 = -3$
   - $-1 \times 2 = -2$
   - $-1 \times 1 = -1$
   - $-1 \times 0 = 0$
   - $-1 \times (-1) = 1$
   - $-1 \times (-2) = 2$
   - $-1 \times (-3) = 3$

   Explain that as the multiplier is decreasing by 1, the product is increasing by 1. The pattern continues and each answer is one more than the one before it. Emphasize that a negative integer multiplied by a negative integer results in a positive product.

2. Review the rules and examples for multiplying integers on the worksheet with your students. If necessary, review absolute values with your students in 1.14: "Finding Absolute Values and Opposites."

EXTRA HELP:

Multiplication is commutative; numbers can be multiplied in any order.

ANSWER KEY:

(1) 32 **(2)** −20 **(3)** 6 **(4)** −18 **(5)** 63 **(6)** −84 **(7)** 80 **(8)** −56 **(9)** −27 **(10)** 60 **(11)** 4 **(12)** −48

(Challenge) Disagree. Explanations may vary. The statement is true sometimes but not always. For example, the statement is true if two integers with the same sign are multiplied or if three positive integers are multiplied. But when three or any odd number of negative integers are multiplied it is not true.

Name _____ Date _____

WORKSHEET 1.17: MULTIPLYING TWO INTEGERS

Follow the steps below to multiply two integers:

1. Multiply the absolute values of the numbers.
2. Determine the sign of the product.
   - If both integers are positive or both are negative, the product is positive.
   - If one integer is positive and the other is negative, the product is negative.
   - If one integer is zero, the product is neither positive nor negative. It is zero.

EXAMPLES

$3 \times (-5) = -15$ $-3 \times (-6) = 18$ $-4 \times 5 = -20$ $5 \times 9 = 45$

DIRECTIONS: Find each product.

1. 8×4 **2.** $5 \times (-4)$ **3.** $-3 \times (-2)$

4. -6×3 **5.** $-7 \times (-9)$ **6.** -12×7

7. $-16 \times (-5)$ **8.** -14×4 **9.** -3×9

10. $-10 \times (-6)$ **11.** $-2 \times (-2)$ **12.** $4 \times (-12)$

CHALLENGE: Do you agree with this statement? The product of two or more integers with the same sign is always positive. Explain your answer.

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Teaching Notes 1.18: Multiplying More Than Two Integers

Students who possess a weak understanding of the rules for multiplying two integers will almost certainly have trouble multiplying more than two integers. The following steps can help reinforce the process.

1. Review the basic rules for multiplying two integers with your students:
   - If both integers are positive or both are negative, the product is positive.
   - If one integer is positive and the other is negative, the product is negative.
   - If one integer is zero, the product is zero. (Zero is neither positive nor negative.)
2. Explain that only two integers can be multiplied at one time. For simplicity, if three or more integers are to be multiplied, the absolute values of the first two should be multiplied and the sign should be determined. Then the absolute value of this product should be multiplied by the absolute value of the next integer and the sign should be determined. The process continues in this manner for additional integers. You may wish to note that because multiplication is commutative, integers may be multiplied in any order.
3. To reinforce how students may determine the correct sign when multiplying more than two integers, emphasize the following:
   - If all of the integers are positive, the product is positive.
   - If there is an odd number of negative integers in the problem, the product is negative.
   - If there is an even number of negative integers in the problem, the product is positive.
   - If zero is a factor, the product is zero no matter how many other factors are multiplied.
4. Review the steps for multiplying and the examples on the worksheet with your students.

EXTRA HELP:

Double-check computation and signs to correct any careless mistakes.

ANSWER KEY:

(1) −144 **(2)** 0 **(3)** 28 **(4)** 36 **(5)** −280 **(6)** −216 **(7)** 180 **(8)** −120 **(9)** −24 **(10)** −27 **(11)** 0 **(12)** 40

(Challenge) It is correct. Explanations may vary. One correct response is that the order of multiplying the factors does not affect the product. There are no mistakes in the answer.

Name _____ Date _____

WORKSHEET 1.18: MULTIPLYING MORE THAN TWO INTEGERS

To multiply more than two integers, follow the steps below:

1. Start with the first two integers and multiply their absolute values.
2. Determine the sign.
   - If the integers are both positive or both negative, the product is positive.
   - If one integer is positive and the other is negative, the product is negative.
3. Multiply the absolute value of the product of the first two integers by the absolute value of the third. Follow the same rules for finding the correct sign. Continue with this process if there are more integers to multiply.
4. If zero is a factor in the problem, the product is zero.

EXAMPLES

$-3 \times (-4) \times (-5) =$	$3 \times (-8) \times (-2) =$	$-4 \times 0 \times 5 =$
$12 \times (-5) =$	$-24 \times (-2) =$	0
-60	48	

DIRECTIONS: Find the products.

1. $6 \times (-8) \times 3$
2. $-15 \times 0 \times 9$
3. $-7 \times (-1) \times 4$
4. $-1 \times (-3) \times 12$
5. $14 \times 5 \times (-4)$
6. $-3 \times (-9) \times (-8)$
7. $15 \times 4 \times 3$
8. $-4 \times (-5) \times 6 \times (-1)$
9. $4 \times (-3) \times 2$
10. $-3 \times (-3) \times (-3)$
11. $-15 \times (-7) \times 0$
12. $5 \times (-4) \times (-2)$

CHALLENGE: Is the procedure used to multiply the following integers correct? Explain your answer. $-3 \times 4 \times (-6) \times 2 = -12 \times (-12) = 144$

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Teaching Notes 1.19: Using Integers as Bases

Raising an integer to a power poses a problem for some students. These students sometimes identify the base of an expression incorrectly and then go on to simplify the expression incorrectly.

1. Explain that $(-3)^2$ is not the same as -3^2.
2. Discuss the meaning of $(-3)^2$. Note that the base, -3, is in parentheses. Explain that $(-3)^2 = -3 \times (-3) = 9$. Because -3 is a factor two times, the product is positive.
3. Discuss the meaning of -3^2. Note that in this case, 3 is the base, not -3. Explain that $-3^2 = -(3 \times 3) = -9$. Because both factors are positive and the negative sign refers to the quantity, the product is -9.
4. Explain that two different expressions may have the same value, for example:
   - $(-3)^3 = -3 \times (-3) \times (-3) = -27$ Because -3 is a factor three times, the product is negative.
   - $-3^3 = -(3 \times 3 \times 3) = -27$ Because the factors are positive and the negative sign refers to the quantity, the product is negative.
5. Review the examples on the worksheet with your students. Encourage your students to pay close attention to the base and write the problems correctly.

EXTRA HELP:

Any number, except 0, raised to the zero power is equal to 1.

ANSWER KEY:

(1) -36 **(2)** 36 **(3)** 125 **(4)** -125 **(5)** -125 **(6)** -1 **(7)** 1 **(8)** -1 **(9)** -8 **(10)** -81 **(11)** -16 **(12)** 64

(Challenge) Yes. Explanations may vary. One response is that an exponent shows the number of times the base is used as a factor. If two negative numbers are multiplied, the product is positive; if three negative numbers are multiplied, the product is negative; and so on.

Name _____ Date _____

WORKSHEET 1.19: USING INTEGERS AS BASES

To simplify an integer raised to a power, you must first identify the base. This is particularly important if the base is a negative number.

EXAMPLES

$(-4)^2$	-4 is the base. 2 is the exponent. $(-4)^2 = -4 \times (-4) = 16$.
-4^2	4 is the base. 2 is the exponent. $-4^2 = -(4 \times 4) = -16$.
$(-4)^3$	-4 is the base. 3 is the exponent. $(-4)^3 = -4 \times (-4) \times (-4) = -64$.
-4^3	4 is the base. 3 is the exponent. $-4^3 = -(4 \times 4 \times 4) = -64$.

DIRECTIONS: Simplify each expression.

1. -6^2 **2.** $(-6)^2$ **3.** 5^3 **4.** -5^3

5. $(-5)^3$ **6.** -1^2 **7.** $(-1)^2$ **8.** $-(-1)^2$

9. -2^3 **10.** $-(3)^4$ **11.** $-(-4)^2$ **12.** $(-8)^2$

CHALLENGE: Is a negative number raised to an odd power always negative? Explain your answer.

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Teaching Notes 1.20: Dividing Integers

Most students have little trouble dividing integers. A trouble spot for some students, however, is not understanding that dividing by zero is not possible.

1. Explain to your students that all division problems can be written in two ways. For example, $-12 \div 3$ is the same as $\frac{-12}{3}$. 3 is the divisor or denominator and -12 is the dividend or numerator. In each case, the answer, of course, is -4.
2. Explain that division can be checked by multiplication. To check that the answer to a division problem is correct, students should multiply the quotient by the divisor. The product should be the same as the dividend. $-12 \div 3 = -4$, therefore $-4 \times 3 = -12$.
3. Offer this example: $0 \div -5 = 0$. This is correct because $0 \times (-5) = 0$.
4. Now offer this example: $-5 \div 0$. Some students may say the answer is 0. Emphasize that 0 is not correct because $0 \times 0 \neq -5$.
5. Explain that you cannot divide by zero. In division, a quantity is divided into groups. But a quantity cannot be divided into zero groups. It is impossible. This is why division by zero is undefined.
6. Review the rules for determining the sign of the quotient and the examples on the worksheet with your students.

EXTRA HELP:

Use multiplication to double-check your work when you divide two integers.

ANSWER KEY:

(1) 10 **(2)** 5 **(3)** 0 **(4)** Undefined **(5)** -11 **(6)** Undefined **(7)** -24 **(8)** 0 **(9)** -9 **(10)** 21 **(11)** 4 **(12)** Undefined

(Challenge) Answers may vary. Following is a possible answer. $0 \div 3 = 0$ but $3 \div 0$ is undefined and has no solution.

Name _____ Date _____

WORKSHEET 1.20: DIVIDING INTEGERS

When dividing integers, determine the sign of the quotient by following the rules below:

1. If both the divisor and the dividend are positive or both are negative, the quotient is positive.
2. If either the divisor or the dividend is positive and the other is negative, the quotient is negative.
3. If 0 is divided by any number, the quotient is 0.
4. If any number is divided by 0, there is no solution because division by zero is undefined.

EXAMPLES

$-72 \div (-8) = 9$ $\quad \frac{-63}{9} = -7$ $\quad 40 \div 5 = 8$ $\quad 0 \div (-12) = 0$ $\quad \frac{-10}{0} =$ undefined

DIRECTIONS: Find the quotients, if possible.

1. $\frac{-100}{-10}$ **2.** $125 \div 25$ **3.** $0 \div (-18)$

4. $\frac{-20}{0}$ **5.** $\frac{-121}{11}$ **6.** $13 \div 0$

7. $144 \div (-6)$ **8.** $\frac{0}{29}$ **9.** $\frac{-108}{12}$

10. $\frac{63}{3}$ **11.** $\frac{-24}{-6}$ **12.** $\frac{-15}{0}$

CHALLENGE: Provide an example that shows that order matters when you divide two integers.

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Teaching Notes 1.21: Finding Absolute Values of Expressions

To simplify an expression written within the absolute value symbol, students must think of the absolute value symbol as a grouping symbol. A common error is finding the absolute value of each number in the absolute value symbol and then finding the sum or difference.

1. Discuss the meaning of absolute value with your students. Point out that $|x|$ is the distance on the number line x is from zero. Depending on the abilities of your students, you may find it helpful to review 1.14: "Finding Absolute Values and Opposites."
2. Explain to your students that they should view absolute value as they would a grouping symbol, simplifying within it first before finding the absolute value of the number or numbers within it.
3. Offer the following examples:
   - $|3 - 12| = |3 + (-12)| = |-9| = 9$
   - $|3| - |12| = 3 - 12 = 3 + (-12) = -9$

   Emphasize that although the expressions look much alike, they are in fact quite different. In the first example, 3 and -12 are grouped within the absolute value symbol. In the second example, 3 and -12 are not grouped together but rather the difference of their absolute values must be found.
4. Review the examples on the worksheet with your students.

EXTRA HELP:

A negative symbol means the opposite. For example, $-|3|$ means the opposite of the absolute value of 3 which is -3.

ANSWER KEY:

(1) 11 **(2)** -18 **(3)** 28 **(4)** 41 **(5)** 84 **(6)** -31 **(7)** 48 **(8)** -42 **(9)** -5 **(10)** 23 **(11)** 10 **(12)** -10

(Challenge) 20 and 40.

Name _____ Date _____

WORKSHEET 1.21: FINDING ABSOLUTE VALUES OF EXPRESSIONS

You can think of the absolute value symbol as a special type of grouping symbol. When working with expressions inside the absolute value symbol, you must find the absolute value of the simplified expression.

EXAMPLES

$\lvert 3 - 10\rvert =$	$\lvert 3(-6)\rvert =$	$\lvert -14 + 20\rvert =$	$3\lvert -15 + 10\rvert =$
$\lvert 3 + (-10)\rvert =$	$\lvert -18\rvert =$	$\lvert 6\rvert =$	$3\lvert -5\rvert =$
$\lvert -7\rvert =$	18	6	$3(5) =$
7			15

DIRECTIONS: Find each absolute value.

1. $\lvert 4 - 15\rvert$

2. $-3\lvert 6 - 12\rvert$

3. $\lvert (-7)(-4)\rvert$

4. $\lvert 15 + 26\rvert$

5. $7\lvert 3(-4)\rvert$

6. $-\lvert 15 - 46\rvert$

7. $\lvert 3(-4)\rvert + \lvert 2(-18)\rvert$

8. $-3\lvert 26 - 12\rvert$

9. $\lvert 5 - 4\rvert - \lvert -6\rvert$

10. $\lvert 3\rvert + 2\rvert - 10\rvert$

11. $\lvert -2 + (-4 \times 2)\rvert$

12. $-2\lvert 1 + 4\rvert$

CHALLENGE: Fill in the blank with a single number so that the following statement is true. There are two possible answers. Try to find both of them.
$\lvert 30 - ____\rvert = 10$

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Teaching Notes 1.22: Finding Square Roots of Square Numbers

When asked to find the square root of a number, some students make the mistake of dividing the number by 2. To remediate this problem, follow the procedure below.

SPECIAL MATERIALS:

Two sheets of graph paper for each student, rulers

1. Instruct your students to draw five squares: a 1×1 square, a 2×2 square, and so on up to a 5×5 square. Ask them to label two adjacent sides of each square and also write the area inside of each square. Ask them to write a number sentence to express the area of each.
2. Explain that the area of each square is equal to the length of a side squared. Thus, the area of a 3×3 square $= 9$ because $3^2 = 9$.
3. Suggest that students think of finding the square root of a number as finding the length of a side of a square. For example, $\sqrt{16} = 4$. 4 is the length of a side of a square that has an area of 16 units. Ask your students to find the square roots for each area they found (in step 1). Identifying these square roots will help them to memorize the process for finding square roots.
4. Review the list of square numbers on the worksheet with your students. Note that each of these numbers has a square root that is a positive integer. Also review the examples.

EXTRA HELP:

Square numbers are always positive.

ANSWER KEY:

(1) 3 **(2)** 6 **(3)** 10 **(4)** 9 **(5)** 1 **(6)** 2 **(7)** -5 **(8)** -1 **(9)** 11 **(10)** 10 **(11)** -3 **(12)** 2

(Challenge) $\sqrt{121} = 11$; $\sqrt{144} = 12$; $\sqrt{169} = 13$

Name _____ Date _____

WORKSHEET 1.22: FINDING SQUARE ROOTS OF SQUARE NUMBERS

Think of a square number as the area of a square. The length of a side is the square root. Following is a list of the first ten square numbers:

1 4 9 16 25 36 49 64 81 100

EXAMPLE

$\sqrt{49}$ is read "the square root of 49."

$\sqrt{49} = 7$ because $7^2 = 49$.

DIRECTIONS: Find each square root.

1. $\sqrt{9}$ **2.** $\sqrt{36}$ **3.** $\sqrt{100}$

4. $\sqrt{81}$ **5.** $\sqrt{1}$ **6.** $\sqrt{4}$

7. $-\sqrt{25}$ **8.** $\sqrt{36} - \sqrt{49}$ **9.** $\sqrt{121}$

10. $\sqrt{64} + \sqrt{4}$ **11.** $-\sqrt{36} + \sqrt{9}$ **12.** $\sqrt{49} - \sqrt{25}$

CHALLENGE: Continue the list of square numbers above by writing the next three square numbers and their square roots.

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SECTION 2

Rational Numbers

Teaching Notes 2.1: Classifying Counting Numbers, Whole Numbers, Integers, and Rational Numbers

Classifying numbers presents problems for some students. Classifying rational numbers can pose special problems.

1. Review the counting numbers, whole numbers, and integers with your students. You may find it helpful to refer to the student worksheet for this.
2. Define rational numbers as any number that can be expressed as the quotient of two integers. Rational numbers can be positive, negative, or zero.
3. Provide examples of rational numbers such as $\frac{3}{5}$, $-\frac{1}{3}$, $-2\frac{1}{4}$, 4, 1.2, and 0.75. Remind your students that whole numbers and decimals can be expressed as fractions.
4. Review the information and examples on the worksheet with your students.

EXTRA HELP:

Being able to classify numbers fosters overall understanding of numbers and their relationships.

ANSWER KEY:

(Counting numbers) 15, 4, 21

(Whole numbers) 15, 0, 4, 21

(Integers) 15, −8, 0, −7, 4, −1, −150, 21

(Rational numbers) All of them

(Challenge) Agree. Explanations may vary. One explanation is that every integer can be expressed as a rational number by writing the integer over 1, such as $\frac{2}{1}$. But fractions and decimals are rational numbers that may or may not be integers, such as $0.\overline{3}$.

Name _____ Date _____

WORKSHEET 2.1: CLASSIFYING COUNTING NUMBERS, WHOLE NUMBERS, INTEGERS, AND RATIONAL NUMBERS

Numbers can be classified into various groups. Four of the most important groups are shown below:

- *Counting numbers:* Starting with 1, the counting numbers continue forever by adding 1 to the previous number: 1, 2, 3, 4, 5, . . .
- *Whole numbers:* Zero and the counting numbers make up the whole numbers: 0, 1, 2, 3, 4, 5, . . .
- *Integers:* The counting numbers, their opposites, and zero make up the integers: . . .−3, −2, −1, 0, 1, 2, 3, . . .
- *Rational numbers:* Any number that can be written as the quotient of two integers make up the rational numbers.

EXAMPLES

$$3, -1, 0, \frac{3}{5}, -\frac{7}{8}, 4.1$$

Many numbers belong to several groups, but some belong to only one group. 5, for example, is a counting number, whole number, integer, and rational number. The fraction $\frac{3}{5}$ can be classified only as a rational number.

DIRECTIONS: Classify each number below.

15	−8	0	−7	4
$1\frac{1}{3}$	$-3\frac{1}{5}$	−1	$\frac{3}{5}$	−150
	0.6	−1.8	21	

CHALLENGE: Do you agree with the following statement? All integers are rational numbers but only some rational numbers are integers. Explain your answer.

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Teaching Notes 2.2: Simplifying Fractions

A common mistake students make when simplifying fractions is to divide by a common factor but not the greatest common factor (GCF). Unless students realize they did not use the GCF, the resulting fraction is not expressed in simplified form.

1. Explain that to simplify a fraction students must divide the numerator and the denominator by the GCF. If your students have trouble finding the GCF, see 1.11: "Finding the Greatest Common Factor."
2. Explain that a fraction is in simplest form if the GCF of the numerator and denominator is 1. For example, $\frac{11}{15}$ is in its simplest form because the GCF of 11 and 15 is 1. Emphasize that students should check that the GCF of the numerator and denominator is 1 to determine if the fraction is in simplest form. If the GCF is not 1, they should divide the numerator and denominator by a common factor, continuing this process until the GCF is 1.
3. Explain that negative fractions are simplified in the same manner as positive fractions. Note that the negative sign may be written before the numerator, before the fraction, or before the denominator. For example, $\frac{-3}{5}$, $-\frac{3}{5}$, and $\frac{3}{-5}$ all express the same number.
4. Review the example on the worksheet with your students.

EXTRA HELP:

If the numerator and denominator of a fraction are even, the fraction is not simplified.

ANSWER KEY:

(1) $\frac{1}{2}$ (2) $-\frac{5}{7}$ (3) $-\frac{4}{5}$ (4) $\frac{1}{5}$ (5) $\frac{6}{13}$ (6) $\frac{2}{3}$ (7) $\frac{8}{11}$ (8) $-\frac{5}{16}$ (9) $-\frac{5}{8}$ (10) $\frac{35}{46}$
(11) $-\frac{3}{7}$ (12) $\frac{4}{5}$

(Challenge) Answers may vary. For example, some students may argue that using 15 as the GCF is indeed the easier way. Others may feel that dividing first by 5 and then by 3 is easier because it is easier to recognize that 5 is a common factor of 105 and 120. $\frac{105}{120} = \frac{21}{24}$ It is then easy to see that 3 is a common factor of 21 and 24. $\frac{21}{24} = \frac{7}{8}$

Name _____ Date _____

WORKSHEET 2.2: SIMPLIFYING FRACTIONS

To simplify a fraction, divide the numerator and denominator by the greatest common factor (GCF). When a fraction is simplified, the GCF of the numerator and denominator is 1.

EXAMPLE

Simplify $\frac{90}{120}$.

The GCF of 90 and 120 is 30. $\frac{90 \div 30}{120 \div 30} = \frac{3}{4}$

Because the GCF of 3 and 4 is 1, the fraction is simplified.

DIRECTIONS: Simplify.

1. $\frac{12}{24}$ **2.** $-\frac{15}{21}$ **3.** $-\frac{28}{35}$ **4.** $\frac{23}{115}$

5. $\frac{18}{39}$ **6.** $\frac{42}{63}$ **7.** $\frac{48}{66}$ **8.** $-\frac{20}{64}$

9. $-\frac{25}{40}$ **10.** $\frac{70}{92}$ **11.** $-\frac{105}{245}$ **12.** $\frac{64}{80}$

CHALLENGE: Sam simplified $\frac{105}{120}$ by using 15 as the GCF. In your opinion, is there an easier way to simplify this fraction? Explain your answer.

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Teaching Notes 2.3: Rewriting Mixed Numbers as Improper Fractions

When rewriting mixed numbers as improper fractions in the lower grades, most students were taught to multiply the denominator by the whole number, add the numerator, and then place the answer over the denominator. This rule applies only to positive mixed numbers, however, and can be the source of errors if applied to negative mixed numbers.

1. Explain why the rule your students learned in the lower grades is valid for rewriting a positive mixed number as an improper fraction. Use the following example: One large pizza is cut into 8 slices. If a student orders $2\frac{1}{8}$ pizzas for his friends and himself, he has 16 slices (two pies), plus 1 more slice for a total of 17 slices. Therefore, $2\frac{1}{8} = \frac{17}{8}$.
2. Explain that the method above does not apply to rewriting negative mixed numbers. Note that a negative number is the opposite of a positive number. To change a negative mixed number to an improper fraction, students must express the negative mixed number as a positive improper fraction and then write its opposite. For example, $-2\frac{1}{8} = -\left(2\frac{1}{8} = \frac{8 \times 2 + 1}{8}\right) = -\frac{17}{8}$.
3. Review the information and examples on the worksheet with your students.

EXTRA HELP:

Before expressing the mixed number as an improper fraction, be sure it is in simplest form.

ANSWER KEY:

(1) $\frac{23}{4}$ **(2)** $-\frac{53}{8}$ **(3)** $\frac{38}{15}$ **(4)** $-\frac{47}{6}$ **(5)** $\frac{43}{12}$ **(6)** $-\frac{39}{4}$ **(7)** $\frac{27}{16}$ **(8)** $-\frac{49}{10}$ **(9)** $\frac{24}{5}$ **(10)** $-\frac{52}{5}$

(Challenge) Yes, they are equivalent but each number could be rewritten. $3\frac{1}{3} = \frac{10}{3}$

Name _____ Date _____

WORKSHEET 2.3: REWRITING MIXED NUMBERS AS IMPROPER FRACTIONS

Every mixed number (positive or negative) can be rewritten as an improper fraction.

EXAMPLES

1. To rewrite a positive mixed number as an improper fraction, multiply the denominator by the whole number, add the numerator, and write the answer over the denominator.

$$3\frac{2}{7} = \frac{7 \times 3 + 2}{7} = \frac{23}{7}$$

2. To rewrite a negative mixed number as an improper fraction, write it as a positive improper fraction, follow the procedure above, and then write its opposite.

$$-3\frac{2}{7} = -\left(3\frac{2}{7}\right) = -\left(\frac{7 \times 3 + 2}{7}\right) = -\frac{23}{7}$$

DIRECTIONS: Rewrite the following mixed numbers as improper fractions.

1. $5\frac{3}{4}$

2. $-6\frac{5}{8}$

3. $2\frac{8}{15}$

4. $-7\frac{5}{6}$

5. $3\frac{7}{12}$

6. $-9\frac{3}{4}$

7. $1\frac{11}{16}$

8. $-4\frac{9}{10}$

9. $4\frac{4}{5}$

10. $-10\frac{2}{5}$

CHALLENGE: Is the following answer correct? $3\frac{15}{45} = \frac{150}{45}$ Explain your reasoning.

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Teaching Notes 2.4: Comparing Rational Numbers

Comparing two rational numbers, especially two negative rational numbers, is a problem for some students. Although they may understand that $\frac{2}{3} > \frac{1}{3}$, they may have trouble understanding that $-\frac{1}{3} > -\frac{2}{3}$.

1. Explain the meaning of $>$ and $<$, emphasizing that the symbol always points to the smaller number. (Or you might say that the symbol opens to the larger number.)
2. Caution your students to pay close attention to negative rational numbers. Point out that although $\frac{1}{2} > \frac{1}{4}$, $-\frac{1}{2} < -\frac{1}{4}$. Similarly, $0.50 > 0.25$ but $-0.50 < -0.25$.
3. Suggest that when comparing rational numbers students express each fraction or mixed number as an equivalent fraction with a positive denominator. They should then compare the numerators. Note that this reduces the chances that they will make a mistake because of the sign of the fraction.
4. Review and discuss the number line, information, and the example on the worksheet with your students. If necessary, review the steps for finding common denominators and writing equivalent fractions.

EXTRA HELP:

A positive number is always larger than a negative number.

ANSWER KEY:

(1) $>$ (2) $>$ (3) $<$ (4) $<$ (5) $>$ (6) $<$

(Challenge) Yes. Explanations may vary. One response is $-\frac{1}{4} = \frac{-5}{20}$ and $-\frac{2}{5} = \frac{-8}{20}$. If the numerators are compared, $-5 > -8$, therefore $-\frac{1}{4} > -\frac{2}{5}$.

Name _____ Date _____

WORKSHEET 2.4: COMPARING RATIONAL NUMBERS

Every rational number can be paired with a point on a number line. A number is larger than any number to its left and smaller than any number to its right.

You can compare rational numbers by using a number line or writing equivalent fractions. After writing equivalent fractions, compare the numerators using $>$ (greater than) or $<$ (less than). Remember to include any negative signs.

EXAMPLE

Compare $-\frac{7}{8}$ and $-\frac{9}{10}$.

$-\frac{7}{8} = \frac{-7 \times 5}{8 \times 5} = \frac{-35}{40}$ $\quad -\frac{9}{10} = \frac{-9 \times 4}{10 \times 4} = \frac{-36}{40}$ $\quad$ Because $-35 > -36$, $-\frac{7}{8} > -\frac{9}{10}$.

DIRECTIONS: Compare each pair of numbers by using $>$ or $<$.

1. $-\frac{1}{3}, -\frac{3}{8}$

2. $\frac{1}{2}, -\frac{3}{5}$

3. $-\frac{1}{8}, \frac{1}{8}$

4. $-\frac{4}{5}, -\frac{6}{10}$

5. $4\frac{4}{5}, 4\frac{3}{4}$

6. $-9\frac{7}{10}, -9\frac{2}{3}$

CHALLENGE: Jillian compared $-\frac{1}{4}$ and $-\frac{2}{5}$ this way: She said that $\frac{1}{4} < \frac{2}{5}$. Because $-\frac{1}{4}$ is the opposite of $\frac{1}{4}$ and $-\frac{2}{5}$ is the opposite of $\frac{2}{5}$, $-\frac{1}{4} > -\frac{2}{5}$. Is her answer correct? Explain.

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Teaching Notes 2.5: Expressing Rational Numbers as Decimals

A common error students make when writing rational numbers as decimals is failing to realize whether a decimal repeats or, if they do realize it repeats, not identifying the correct sequence of repeating digits. Negative numbers can add to the confusion.

1. Explain that all rational numbers can be represented as a terminating decimal or a repeating decimal. Note that a terminating decimal has no remainder and that a repeating decimal will always have a remainder. A repeating decimal will also always have a number or group of numbers that repeat. (You might want to mention that, although they are not rational numbers, there are decimals that are nonterminating and nonrepeating. These are called irrational numbers. The number π is an example.)
2. Review the steps for expressing rational numbers as decimals and the examples on the worksheet with your students. Depending on their abilities, you may find it necessary to show the steps for long division. Remind your students that they must place a bar over the repetend, which is the number or numbers that repeat.
3. Emphasize that for negative fractions students must use the fraction's absolute value when dividing the numerator by the denominator. They must then remember to place the negative sign in their answer. Depending on the abilities of your students, you may find it helpful to review absolute value. See 1.14: "Finding Absolute Values and Opposites."

EXTRA HELP:

Calculators may express fractions as decimals incorrectly. Some calculators either round the decimal to a given place or omit digits after a certain place.

ANSWER KEY:

(1) 0.16 **(2)** $-0.\overline{6}$ **(3)** $-0.1\overline{6}$ **(4)** $-0.\overline{4}$ **(5)** 0.45 **(6)** 0.1875 **(7)** $-0.41\overline{6}$ **(8)** $0.\overline{428571}$ **(9)** $0.\overline{90}$ **(10)** $-0.\overline{1}$ **(11)** $0.\overline{714285}$ **(12)** -0.25

(Challenge) $0.\overline{3}$ is greater than 0.3. Explanations may vary. A possible explanation is that 0.3 is equal to $\frac{3}{10}$, and that $0.\overline{3}$ is equal to $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} \ldots$.

Name _____ Date _____

WORKSHEET 2.5: EXPRESSING RATIONAL NUMBERS AS DECIMALS

A fraction is a rational number. Follow the steps below to express fractions as decimals:

1. For positive fractions, divide the numerator by the denominator. Add a decimal point and a zero after the numerator. Add more zeros if necessary. Keep dividing until the remainder is zero or until a number or group of numbers repeats in the quotient.
   - A remainder of zero results in a terminating decimal.
   - A decimal in which a number or group of numbers repeats is called a "nonterminating decimal." The repeating number or numbers are known as a "repetend" and must be written with a bar over them.
2. For negative fractions, divide the absolute value of the numerator by the absolute value of the denominator. Then follow the procedure above. Place the negative sign in the decimal.

EXAMPLES

Express $\frac{5}{8}$ and $-\frac{5}{6}$ as decimals.

$$\frac{5}{8} \rightarrow 8\overline{)5.000}\ 0.625 \rightarrow \frac{5}{8} = 0.625 \qquad \frac{5}{6} \rightarrow -6\overline{)5.000}\ -0.833 \rightarrow -\frac{5}{6} = -0.8\overline{3}$$

DIRECTIONS: Express each rational number as a repeating or terminating decimal.

1. $\frac{4}{25}$ **2.** $-\frac{2}{3}$ **3.** $-\frac{1}{6}$ **4.** $-\frac{4}{9}$

5. $\frac{9}{20}$ **6.** $\frac{3}{16}$ **7.** $-\frac{5}{12}$ **8.** $\frac{3}{7}$

9. $\frac{10}{11}$ **10.** $-\frac{2}{18}$ **11.** $\frac{10}{14}$ **12.** $-\frac{3}{12}$

CHALLENGE: How does $0.\overline{3}$ compare with 0.3? Explain your answer.

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Teaching Notes 2.6: Expressing Terminating Decimals as Fractions or Mixed Numbers

Expressing terminating decimals as fractions is a relatively straightforward procedure. Two common errors include using an incorrect decimal place in writing the fraction and failing to use the negative sign with negative numbers.

1. Explain that a terminating decimal is a decimal that does not repeat. For example, 0.5, 0.894, and −2.35 are terminating decimals.
2. Explain that terminating decimals can be expressed as fractions or mixed numbers. (You might mention that repeating decimals can also be expressed as fractions or mixed numbers.)
3. Review the place value chart and examples on the worksheet with your students. Explain that they must consider the number of decimal places to the right of the decimal point when they write a decimal as a fraction. Make sure that your students fully understand place value. A decimal that ends in the tenths place will equal a fraction with a denominator of 10; a decimal that ends in the hundredths place will equal a fraction with a denominator of 100; a decimal that ends in the thousandths place will equal a fraction with a denominator of 1,000; and so on.
4. Remind your students that a negative decimal must be expressed as a negative fraction and that fractions should always be simplified.

EXTRA HELP:

You can always double-check your answer by changing the fraction or mixed number to a decimal.

ANSWER KEY:

(1) $\frac{39}{50}$ **(2)** $-\frac{3}{10}$ **(3)** $-1\frac{7}{100}$ **(4)** $15\frac{33}{40}$ **(5)** $-6\frac{4}{25}$ **(6)** $-\frac{1}{40}$ **(7)** $18\frac{3}{4}$ **(8)** $-20\frac{1}{100}$

(Challenge) There are two errors. $-3\frac{15}{100} \neq 3\frac{5}{20}$. The fraction is simplified incorrectly and the negative sign is missing. The correct answer is $-3\frac{15}{100} = -3\frac{3}{20}$.

Name _____ Date _____

WORKSHEET 2.6: EXPRESSING TERMINATING DECIMALS AS FRACTIONS OR MIXED NUMBERS

Every terminating decimal can be expressed as a fraction or mixed number. Study the place value chart for the number 21,065.4897.

Ten thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-thousandths
2	1	0	6	5	.	4	8	9	7

EXAMPLES

0.4 is read "four tenths." $0.4 = \frac{4}{10} = \frac{2}{5}$

-1.35 is read "negative one and thirty-five hundredths." $-1.35 = -1\frac{35}{100} = -1\frac{7}{20}$

3.105 is read "three and one hundred five thousandths." $3.105 = 3\frac{105}{1000} = 3\frac{21}{200}$

DIRECTIONS: Express each decimal as a fraction or mixed number. Simplify if possible.

1. 0.78 **2.** -0.3 **3.** -1.07 **4.** 15.825

5. -6.16 **6.** -0.025 **7.** 18.75 **8.** -20.01

CHALLENGE: Find the error or errors in this problem. $-3.15 = -3\frac{15}{100} = 3\frac{5}{20}$

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Teaching Notes 2.7: Expressing Repeating Decimals as Fractions or Mixed Numbers

The process of expressing a repeating decimal as a fraction requires students to write an equation, rewrite the decimal, and solve an equation. Confusion with any of these steps can lead to mistakes.

1. Explain that there are two types of nonterminating decimals: those whose digits repeat in a specific pattern and those whose digits do not repeat in a specific pattern.
2. Explain that if the digits repeat in a specific pattern, a bar can be placed over the repeating digits and the decimal can be expressed as a fraction. For example, $0.777\ldots = 0.\overline{7}$ and $0.313131\ldots = 0.\overline{31}$. Unlike terminating decimals such as 0.7, which equals $\frac{7}{10}$ and 0.31, which equals $\frac{31}{100}$, the repeating decimals cannot be written over a power of 10. Also, emphasize that if $n = 0.\overline{7}$, then $10n = 7.\overline{7}$. Similarly, if $n = 0.\overline{31}$, then $100n = 31.\overline{31}$. An example of a nonterminating decimal whose digits do not repeat in a specific pattern is π.
3. Review the information and example on the worksheet with your students. Explain that they should find two equations so that when one equation is subtracted from the other, a terminating decimal is in the equation that results. Note that multiplying an equation by any nonzero number produces an equation that has the same solution as the original equation. Also note that multiplying by 10 "moves" the decimal point one place to the right; multiplying by 100 "moves" the decimal two places to the right, and so on.

EXTRA HELP:

Check your answer by dividing the numerator by the denominator to see if the quotient is the same as the repeating decimal.

ANSWER KEY:

(1) $\frac{1}{9}$ **(2)** $\frac{23}{99}$ **(3)** $\frac{8}{11}$ **(4)** $\frac{5}{9}$ **(5)** $2\frac{9}{11}$ **(6)** $\frac{1}{33}$ **(7)** $6\frac{512}{999}$ **(8)** $7\frac{6}{37}$

(Challenge) Manny is correct. Roseanne's example of $0.\overline{3} = \frac{1}{3}$ is equivalent to $\frac{3}{9}$.

Name _____ Date _____

WORKSHEET 2.7: EXPRESSING REPEATING DECIMALS AS FRACTIONS OR MIXED NUMBERS

Repeating decimals are represented by placing a bar over the digit or digits that repeat. All repeating decimals can be represented as fractions. Follow the steps below:

1. Write an equation that states $n =$ the repeating decimal.
2. Multiply by a power of 10, depending on the number of digits that repeat. (If one digit repeats, multiply by 10; if two digits repeat, multiply by 100; if three digits repeat, multiply by 1,000; and so on.)
3. Subtract the original equation from the equation you wrote.
4. Solve the equation and simplify the answer.

EXAMPLE

Express $0.\overline{75}$ as a fraction.

$$n = 0.\overline{75}$$

Because two digits repeat, multiply the equation by 100. $100n = 75.\overline{75}$

Subtract. $\begin{aligned} 100n &= 75.\overline{75} \\ -n &= 0.\overline{75} \\ \hline 99n &= 75 \end{aligned}$

Solve the equation. $n = \frac{75}{99} = \frac{25}{33}$

DIRECTIONS: Express each decimal as a fraction or mixed number.

1. $0.\overline{1}$ **2.** $0.\overline{23}$ **3.** $0.\overline{72}$ **4.** $0.\overline{5}$

5. $2.\overline{81}$ **6.** $0.\overline{03}$ **7.** $6.\overline{512}$ **8.** $7.\overline{162}$

CHALLENGE: Manny said that he found a shortcut to use when one digit repeats: Just write the number over 9. He supported this statement with the following examples: $0.\overline{2} = \frac{2}{9}$ and $0.\overline{4} = \frac{4}{9}$. Roseanne said that his shortcut is not valid because $0.\overline{3} = \frac{1}{3}$, which does not have a denominator of 9. Who is correct?

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Teaching Notes 2.8: Adding Rational Numbers

Adding rational numbers requires mastery of several skills: finding the least common denominator, writing equivalent fractions, applying the rules for adding integers, and simplifying the result. Depending on the problem, students may also have to change a mixed number to an improper fraction or change an improper fraction to a mixed number. Negative numbers can make the process even more confusing.

1. Explain that a rational number is a number that can be expressed as a quotient of integers. Numbers such as $\frac{1}{4}$, $-\frac{5}{2}$, and 8 are examples of rational numbers.
2. Explain that adding positive and negative rational numbers involves many of the same steps that are necessary for adding positive fractions. Depending on the abilities of your students, you may find it helpful to review the steps for finding a common denominator, writing equivalent fractions, adding the numerators, and simplifying the results.
3. Review the steps for adding rational numbers and the example on the worksheet with your students. Emphasize that to add negative rational numbers, students must also do the following:
   - Write all mixed numbers as improper fractions.
   - Write the denominators of the fractions as positive numbers.
   - Add the numerators, following the rules for adding integers.

EXTRA HELP:

Pay close attention to negative signs as you add rational numbers. It is a good idea to write negative signs before the numerators.

ANSWER KEY:

(1) $-\frac{19}{20}$	**(2)** $\frac{27}{56}$	**(3)** $2\frac{5}{24}$	**(4)** $-2\frac{5}{14}$	**(5)** $-4\frac{11}{20}$
(6) $-\frac{1}{8}$	**(7)** $1\frac{17}{28}$	**(8)** $-6\frac{19}{30}$	**(9)** $-\frac{15}{56}$	**(10)** $-1\frac{7}{10}$

(Challenge) Answers may vary. 1 and $-\frac{1}{2}$ is one solution.

Name _____ Date _____

WORKSHEET 2.8: ADDING RATIONAL NUMBERS

Follow the steps below to add positive and negative rational numbers:

1. Express any mixed or whole numbers as improper fractions.
2. Rewrite any negative fractions so that the denominator is positive.
3. Find the least common denominator and write equivalent fractions.
4. Add the numerators, following the rules for adding integers.
5. Write the sum over the denominators.
6. Simplify the answer.

EXAMPLE

$$-4\frac{2}{3}+1\frac{5}{6}=\frac{-14}{3}+\frac{11}{6}=\frac{-14\times 2}{3\times 2}+\frac{11}{6}=\frac{-28}{6}+\frac{11}{6}=\frac{-17}{6}=-2\frac{5}{6}$$

DIRECTIONS: Find each sum.

1. $-\frac{1}{5}+\left(-\frac{3}{4}\right)$

2. $\frac{5}{8}+\left(-\frac{1}{7}\right)$

3. $-\frac{1}{8}+2\frac{1}{3}$

4. $-5+2\frac{9}{14}$

5. $-2\frac{3}{4}+\left(-1\frac{4}{5}\right)$

6. $\frac{3}{4}+\left(-\frac{7}{8}\right)$

7. $3\frac{6}{7}+\left(-2\frac{1}{4}\right)$

8. $-2\frac{4}{5}+\left(-3\frac{5}{6}\right)$

9. $\frac{6}{7}+\left(-1\frac{1}{8}\right)$

10. $-6\frac{1}{10}+4\frac{2}{5}$

CHALLENGE: Find a positive rational number and a negative rational number whose sum is $\frac{1}{2}$.

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Teaching Notes 2.9: Subtracting Rational Numbers

When students subtract rational numbers, they must remember to rewrite the subtraction problem as an addition problem and then follow the rules for adding rational numbers. The several steps involved in this process can result in frequent mistakes.

1. Explain that a rational number is a number that can be expressed as a quotient of integers. Numbers such as 3, -7, $-2\frac{1}{4}$, $5\frac{2}{3}$, and $\frac{7}{9}$ are examples of rational numbers.
2. Explain that when subtracting a rational number from another rational number, students must rewrite the problem by adding the opposite of the number after the subtraction sign. They must keep the sign of the first number the same. Then they must follow the rules for adding rational numbers. Depending on the abilities of your students, you may wish to reinforce these rules with subtraction of integers. Once your students become familiar with the process, they can move onto subtraction with more complicated rational numbers.
3. Review the steps for subtracting rational numbers and the example on the worksheet. Depending on the abilities of your students, you may find it helpful to also review the steps for finding a common denominator, writing equivalent fractions, adding the numerator, and simplifying the result.

EXTRA HELP:

Be sure to rewrite the problem correctly before you begin to solve the problem.

ANSWER KEY:

(1) $-\frac{17}{20}$ **(2)** $\frac{23}{24}$ **(3)** $2\frac{19}{20}$ **(4)** $-4\frac{1}{5}$ **(5)** $-\frac{1}{2}$ **(6)** $-\frac{1}{8}$ **(7)** $5\frac{17}{28}$

(8) $-2\frac{11}{20}$ **(9)** $-2\frac{13}{20}$ **(10)** $1\frac{59}{78}$

(Challenge) Answers may vary. One correct problem is $-2\frac{1}{3} - 5\frac{1}{2}$. $-2\frac{1}{3} + \left(-5\frac{1}{2}\right) = -7\frac{5}{6}$

Name _____ Date _____

WORKSHEET 2.9: SUBTRACTING RATIONAL NUMBERS

Follow the steps below to subtract positive and negative rational numbers:

1. Rewrite the problem by adding the opposite of the number after the subtraction sign.
2. Express any mixed or whole numbers as improper fractions.
3. Rewrite any negative fractions so that the denominators are positive.
4. Find the least common denominator and write equivalent fractions.
5. Add numerators, following the rules for adding integers.
6. Write the sum over the denominator.
7. Simplify the answer.

EXAMPLE

$$2\frac{1}{8} - 3\frac{4}{5} = 2\frac{1}{8} + \left(-3\frac{4}{5}\right) = \frac{17}{8} + \frac{-19}{5} = \frac{85}{40} + \frac{-152}{40} = \frac{-67}{40} = -1\frac{27}{40}$$

DIRECTIONS: Find the difference.

1. $-\frac{1}{4} - \frac{3}{5}$

2. $\frac{1}{3} - \left(-\frac{5}{8}\right)$

3. $-4\frac{3}{10} - \left(-7\frac{1}{4}\right)$

4. $3 - 7\frac{1}{5}$

5. $4\frac{3}{4} - 5\frac{1}{4}$

6. $\frac{1}{4} - \frac{3}{8}$

7. $4\frac{3}{4} - \left(-\frac{6}{7}\right)$

8. $-1\frac{4}{5} - \frac{3}{4}$

9. $-\frac{9}{10} - 1\frac{3}{4}$

10. $\frac{12}{13} - \left(-\frac{5}{6}\right)$

CHALLENGE: Write a subtraction problem with rational numbers that can be rewritten so that two negative rational numbers are added. Then solve your problem.

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Teaching Notes 2.10: Multiplying and Dividing Rational Numbers

When students multiply and divide positive rational numbers, they do not need to be concerned whether the sign of the answer is positive or negative. It is positive. However, when multiplying or dividing negative rational numbers, they must determine the sign of their answer. For some students, determining the correct sign can be a problem.

1. Explain that the procedure for multiplying and dividing negative rational numbers is similar to multiplying and dividing positive rational numbers. However, the answers may be positive or negative.
2. Review the procedure for multiplying and dividing rational numbers and the examples on the worksheet with your students. Note that they must pay particular attention to negative signs. Emphasize that when dividing, they must multiply by the reciprocal of the number after the division sign. Be sure that your students understand the steps for multiplying integers and remind them that zero times a number is zero and that a number divided by zero is undefined.

EXTRA HELP:

Double-check the sign of your answer to make sure that it is correct.

ANSWER KEY:

(1) $\frac{5}{9}$ **(2)** $7\frac{1}{2}$ **(3)** $-3\frac{25}{32}$ **(4)** $-\frac{8}{11}$ **(5)** Undefined

(Challenge) It is incorrect. For example, Sarah would rewrite $\frac{3}{4} \div \left(-\frac{1}{2}\right)$ as $\frac{3}{4} \times \left(-\frac{1}{2}\right) = -\frac{3}{8}$. She would then find the reciprocal of her answer, which would be $-\frac{8}{3} = -2\frac{2}{3}$. The correct answer is $\frac{3}{4} \div \left(-\frac{1}{2}\right) = \frac{3}{4} \times \left(-\frac{2}{1}\right) = -\frac{6}{4} = -1\frac{1}{2}$.

Name _____ Date _____

WORKSHEET 2.10: MULTIPLYING AND DIVIDING RATIONAL NUMBERS

To multiply and divide rational numbers, follow the steps below:

1. Write any mixed or whole numbers as improper fractions.
2. Rewrite a division problem by multiplying by the reciprocal of the number after the division sign.
3. Rewrite all fractions so that the denominators are positive.
4. Multiply the numerators following the rules for multiplying integers.
   - The product of two positive or two negative integers is positive.
   - The product of a negative and positive integer is negative.
   - The product of a number and zero is zero.
5. Multiply the denominators.
6. Simplify the answer either before you multiply or after you multiply.

EXAMPLES

$$-1\frac{1}{8}\cdot 2\frac{3}{5}=\frac{-9}{8}\cdot\frac{13}{5}=\frac{-117}{40}=-2\frac{37}{40}$$

$$-1\frac{3}{4}\div\left(-\frac{5}{8}\right)=-\frac{7}{4}\div\left(-\frac{5}{8}\right)=\frac{-7}{4}\cdot\frac{-8}{5}=\frac{56}{20}=2\frac{16}{20}=2\frac{4}{5}$$

DIRECTIONS: Find each product or quotient.

1. $-\frac{5}{6}\cdot\left(-\frac{2}{3}\right)$
2. $-4\frac{1}{2}\div\left(-\frac{3}{5}\right)$
3. $-2\frac{3}{4}\cdot 1\frac{3}{8}$
4. $-4\div 5\frac{1}{2}$
5. $-\frac{3}{8}\div\frac{0}{4}$

CHALLENGE: Sarah may have found a new way for dividing fractions! Instead of rewriting the division problem by multiplying by the reciprocal of the number after the division sign, she simply multiplied the fractions. Then she found the reciprocal of her answer. Is this correct? Explain your answer and provide an example.

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Teaching Notes 2.11: Expressing Large Numbers in Scientific Notation

Scientific notation is a type of mathematical shorthand for writing very large or very small numbers without copying numerous zeros. Students often make two errors when writing large numbers in scientific notation: either they place the decimal point incorrectly or they use an incorrect power of ten.

1. Explain that scientific notation is a way of writing very large (or small) numbers as the product of a decimal between 1 and 10 and a power of 10. This eliminates the need for copying a lot of zeros.
2. Make sure that your students understand that the decimal point may be placed to the right of any whole number. For example, for 152 the decimal point is understood to be after the 2.
3. Explain that to write large numbers in scientific notation, students should place the decimal point to the right of the number written in standard form. They should then move the decimal point as far to the left as possible, placing it after the first nonzero digit. Next they should count the number of places they moved the decimal point. For example, for the number 4,230,000,000, 4 is the first nonzero digit in standard form. Because nine digits are to the right of 4, 9 is the exponent. Written in scientific notation, $4{,}230{,}000{,}000 = 4.23 \times 10^9$.
4. Note that all zeros, except those between two nonzero digits, should be dropped.
5. Review the steps for writing scientific notation and the examples on the worksheet with your students.

EXTRA HELP:

Check that your expressions in scientific notation are numbers between 1 and 10 and that the exponent of 10 indicates the correct number of places the decimal point was moved.

ANSWER KEY:

(1) 3.4×10^5 **(2)** 1.86×10^8 **(3)** 2.405×10^4 **(4)** 1.672×10^4 **(5)** 2.0713×10^6
(6) 4.81×10^3 **(7)** 9.0001×10^5 **(8)** 7.1005×10^7 **(9)** 5.142×10^6 **(10)** 1.1×10^3
(11) 7.0012×10^8 **(12)** 8.915×10^7

(Challenge) Sunil is incorrect because the number 23.5 is not between 1 and 10. The correct answer is 2.35×10^4.

Name _____ Date _____

WORKSHEET 2.11: EXPRESSING LARGE NUMBERS IN SCIENTIFIC NOTATION

Scientific notation is a way of writing numbers as the product of a decimal between 1 and 10 and a power of 10. Follow the steps below to write large numbers in scientific notation:

1. Place the decimal point to the right of the first nonzero digit in standard form.
2. Write the digits, excluding zeros unless a zero is between two nonzero digits.
3. Count the number of places you moved the decimal point from the right. Use that number as the power of 10.

EXAMPLES

$35{,}200 = 3.52 \times 10^4$ The decimal point was moved four places from the right.

$3{,}590{,}000 = 3.59 \times 10^6$ The decimal point was moved six places from the right.

$170{,}400 = 1.704 \times 10^5$ The decimal point was moved five places from the right.

DIRECTIONS: Express each number in scientific notation.

1. 340,000 **2.** 186,000,000 **3.** 24,050

4. 16,720 **5.** 2,071,300 **6.** 4,810

7. 900,010 **8.** 71,005,000 **9.** 5,142,000

10. 1,100 **11.** 700,120,000 **12.** 89,150,000

CHALLENGE: Sunil wrote 23,500 as 23.5×10^3 in scientific notation. Is he correct? Explain your answer.

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Teaching Notes 2.12: Evaluating Rational Expressions

Evaluating expressions when variables represent rational numbers involves the application of the order of operations. Students must follow the order of operations correctly and they must be able to add, subtract, multiply, and divide rational numbers correctly. Mistakes may occur anywhere within the procedure.

1. Explain that to evaluate an expression, students must substitute the correct values for variables and then follow the rules for the order of operations.
2. Depending on the abilities of your students, you may find it helpful to review the process for adding, subtracting, multiplying, and dividing rational numbers.
3. Review the steps of the order of operations and the example on the worksheet with your students. Remind them of basic notation:
   - $-a$ means the opposite of a, so if $a = -\frac{3}{4}$, $-a = \frac{3}{4}$.
   - ab means $a \times b$.
   - $\frac{a}{b}$ means $a \div b$.

EXTRA HELP:

Be sure that you substitute the correct values for every variable in an expression.

ANSWER KEY:

(1) 2 **(2)** $1\frac{1}{8}$ **(3)** $-\frac{2}{3}$ **(4)** $\frac{1}{18}$ **(5)** 1 **(6)** $-\frac{4}{9}$

(7) $\frac{1}{2}$ **(8)** $\frac{2}{3}$ **(9)** $-\frac{7}{12}$ **(10)** $\frac{4}{9}$ **(11)** $-\frac{1}{64}$ **(12)** 6

(Challenge) $a = \frac{1}{5}$ and $a = -\frac{1}{5}$ because $\left(\frac{1}{5}\right)^2 = \frac{1}{25}$ and $\left(-\frac{1}{5}\right)^2 = \frac{1}{25}$.

Name _____ Date _____

WORKSHEET 2.12: EVALUATING RATIONAL EXPRESSIONS

Follow the steps below to evaluate expressions:

1. Rewrite the expression by replacing all of the variables with their values.
2. Follow the order of operations.
   - Simplify expressions within grouping symbols first. If there are nested grouping symbols, simplify the innermost first and then work outward.
   - Simplify powers.
   - Multiply and divide in order from left to right.
   - Add and subtract in order from left to right.

EXAMPLE

If $a = \frac{1}{2}, b = -\frac{2}{3}, c = -\frac{1}{3}$, find $a(b + c)$.

$$a(b+c) = \frac{1}{2}\left(\frac{-2}{3} + \frac{-1}{3}\right) = \frac{1}{2}\left(\frac{-3}{3}\right) = \frac{1}{2}(-1) = -\frac{1}{2}$$

DIRECTIONS: Evaluate each expression if $a = \frac{1}{2}, b = -\frac{2}{3}, c = \frac{3}{4}, d = -\frac{1}{3}$.

1. $a + 2c$ **2.** $3ac$ **3.** $4(a + b)$ **4.** $(a + b)(b - d)$

5. $4a^2$ **6.** $-b^2$ **7.** $ad - b$ **8.** $\frac{a}{c}$

9. $a + b - c - d$ **10.** $-\frac{ab}{c}$ **11.** $(cd)^3$ **12.** $-\frac{2}{d}$

CHALLENGE: $a^2 = \frac{1}{25}$. What values of a would make this statement true? Explain your answer.

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Teaching Notes 2.13: Writing Ratios Correctly

When students write ratios, they may inadvertently switch the order of the terms, use a wrong label, or use faulty data. Each of these mistakes will result in writing an incorrect ratio.

1. Explain to your students that order matters when working with ratios. The ratio of 3 to 4 is different from the ratio of 4 to 3. The first quantity must be the first term; the second quantity must be the second term.
2. Explain that ratios must be written in the same units or measurements. If one unit of measure can be expressed in terms of another, students should rewrite the larger in terms of the smaller. For example, because 1 yard equals 36 inches, the ratio of 3 inches to 1 yard can be rewritten as 3 inches to 36 inches or $\frac{3}{36}$ or $\frac{1}{12}$. In this example, caution your students not to make the mistake of writing the ratio as 3 to 1, which is incorrect because the units—inches and yards—are different.
3. Review the information and examples on the worksheet with your students. Remind them to make certain that they are using the correct information when writing ratios.

EXTRA HELP:

Be sure that ratios are expressed in simplest form.

ANSWER KEY:

(1) $\frac{3}{4}$ (2) $\frac{20}{1}$ (3) $\frac{2}{3}$ (4) $\frac{2}{1}$ (5) $\frac{1}{4}$ (6) $\frac{1}{100}$ (7) $\frac{2}{5}$ (8) $\frac{1}{20}$

(Challenge) Because 15 is the least common denominator of 3 and 15, $\frac{2}{3}$ can be written as $\frac{10}{15}$. This means that the team won 10 games if they played 15 games because $\frac{10}{15}$ is the ratio of games won to games played. Because the problem states the ratio of games lost to games played is $\frac{4}{15}$, the team lost 4 games. If they played 15 games, won 10, and lost 4, they tied 1 game. The ratio of games tied to games played is $\frac{1}{15}$. If a larger common denominator was used, the ratio of games tied to games played would be a ratio equivalent to $\frac{1}{15}$.

Name _____ Date _____

WORKSHEET 2.13: WRITING RATIOS CORRECTLY

A ratio is a pair of numbers that makes a comparison or describes a rate. $\frac{a}{b}$, $a{:}b$, or a to b are three ways to express a ratio. a is the first term of the ratio and b is the second term.

EXAMPLES

1. Write the ratio of the number of sides of a triangle to the number of sides of a square. Because order matters, the number of sides of a triangle, 3, is the first term. The number of sides of a square, 4, is the second term. The ratio is $\frac{3}{4}$.
2. Find the ratio of an inch to a foot. Because 1 foot equals 12 inches, the ratio of an inch to a foot is $\frac{1}{12}$.

DIRECTIONS: Write a ratio to describe each situation. Express each ratio in simplest form.

1. Reading 3 books in 4 weeks
2. Traveling 60 miles in 3 hours
3. 2 feet to 1 yard
4. The diameter of a circle to the radius
5. 5 computers to 20 students
6. A penny to a dollar
7. 12 girls to 30 students
8. 5 centimeters to a meter

CHALLENGE: The ratio of Simone's soccer team's games won to games played is $\frac{2}{3}$ and the ratio of games lost to games played was $\frac{4}{15}$. Write the ratio of games tied to games played. Explain your answer.

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Teaching Notes 2.14: Writing and Solving Proportions

A common error in writing and solving proportions occurs when students set up the proportion incorrectly. They then compare incorrect items or rates.

1. Explain that a proportion is a statement that two ratios are equal. In order to do this, the units in the first term of each ratio must be the same. Likewise, the units in the second term must also be the same. For example, if in the first ratio apples are compared to oranges, apples must be compared to oranges in the second. If apples are compared to oranges in the first and oranges are compared to apples in the second, the proportion is set up incorrectly.
2. Review the example on the worksheet with your students. Note how the ratios are written. Direct your students' attention to the units and emphasize that each ratio is almonds to ounces. To avoid mistakes in setting up a proportion, encourage your students to write the units of the ratios first and then write the numbers.
3. Depending on the abilities of your students, you may find it helpful to review solving proportions. You may also want to review rounding.

EXTRA HELP:

Always double-check that each ratio in the proportion is a comparison of the correct units.

ANSWER KEY:

(1) 900 words; $\frac{60}{1} = \frac{x}{15}$ **(2)** \$0.15; $\frac{20}{2.99} = \frac{1}{x}$ **(3)** 48; $\frac{4}{96} = \frac{x}{1152}$

(4) 26.35 minutes; $\frac{8.5}{1} = \frac{x}{3.1}$

(Challenge) Yes. If Luis reads at a rate of 250 words per minute, he could read 3,750 words in fifteen minutes. $\frac{250}{1} = \frac{x}{15}, x = 3,750$. Therefore he would be able to finish reading a 3,650-word article in fifteen minutes.

Name _____ Date _____

WORKSHEET 2.14: WRITING AND SOLVING PROPORTIONS

A proportion is a statement that two ratios are equal. The units of the first term of each ratio must be the same and the units of the second term of each ratio must be the same.

EXAMPLE

A 6-ounce can of almonds contains 148 almonds. About how many almonds would you expect to find in a 10-ounce can? To write this proportion, you must compare almonds to ounces.

$\frac{\text{almonds}}{\text{ounces}} \rightarrow \frac{148}{6} = \frac{x}{10}$ In each ratio, almonds are the first term and ounces are the second.

$$\begin{aligned} 148 \times 10 &= 6x \\ 1480 &= 6x \\ x &= 246\frac{2}{3} \\ x &\approx 247 \end{aligned}$$

DIRECTIONS: Write and solve proportions for each situation.

1. Bekka can type about sixty words per minute. At that rate, how many words can she type in fifteen minutes?

2. Twenty tea bags cost $2.99. Find the cost of one tea bag. Round your answer to the nearest cent.

3. At a factory, 4 out of every 96 CD cases were defective. At that rate, how many CD cases would you expect to be defective if 1,152 cases were produced?

4. Sue ran a 5-km (3.1-mi) race at an average rate of 8.5 minutes per mile. How long did it take her to run the race?

CHALLENGE: Luis can read about 250 words per minute. Would he be able to read a 3,650-word article in fifteen minutes? Explain your answer.

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Teaching Notes 2.15: Expressing Fractions as Percents

Fractions can be expressed as percents via two different methods: changing the fraction to a decimal and then expressing the percent or writing an equivalent fraction and then expressing the fraction as a percent. Most mistakes result from not understanding the steps to the methods.

1. Explain that a percent is a number that represents a rate per hundred. Percents, therefore, are always based on a hundred. For example, if 60 out of 100 eighth-graders prefer pizza for lunch, 60 percent prefer pizza.
2. Review the two methods for changing fractions to percents and the examples on the worksheet with your students.
   - Point out that in method 1, there must be a digit in both the tenths and hundredths place. Depending on the abilities of your students, you may find it helpful to show the steps of long division.
   - Suggest that students may prefer to use method 2 when the denominator of the fraction is a factor of 100 because it is then easier to write the equivalent fraction. Depending on your students' abilities, you may find it helpful to review solving proportions.
   - Note the examples of changing $\frac{3}{8}$ to a percent. In method 1, the remainder is written as a fraction. In method 2, the remainder is written in decimal form. Both are correct.

EXTRA HELP:

Mixed numbers can be changed to percents that are greater than 100 percent by changing the mixed number to an improper fraction and using method 1 or method 2.

ANSWER KEY:

(1) 60% **(2)** $87\frac{1}{2}\%$ or 87.5% **(3)** $66\frac{2}{3}\%$ or $66.\overline{6}\%$ **(4)** $\frac{7}{10}\%$ or 0.7% **(5)** 9% **(6)** $16\frac{2}{3}\%$ or $16.\overline{6}\%$

(7) 67% **(8)** 350% **(9)** $90.\overline{90}\%$ **(10)** $77.\overline{7}\%$ **(11)** 262.5% **(12)** $8.\overline{3}\%$

(Challenge) Disagree. It should read that whole numbers greater than 1 and all mixed numbers can be expressed as percents greater than 1. 1 is equal to 100 percent.

Name _____ Date _____

WORKSHEET 2.15: EXPRESSING FRACTIONS AS PERCENTS

Every fraction can be expressed as a percent using the methods shown below:

Method 1: Change the fraction to a decimal and express the decimal as a percent.

1. Divide the numerator by the denominator to the hundredths place.
2. If there is a remainder, place it over the divisor.
3. Move the decimal point in the quotient two places to the right and include the percent symbol.

EXAMPLE

$$\frac{1}{5} \rightarrow 5\overline{)1.00}^{0.20} \rightarrow \frac{1}{5} = 20\% \qquad \frac{3}{8} \rightarrow 8\overline{)3.00}^{0.37\frac{4}{8}} \rightarrow \frac{3}{8} = 37\frac{1}{2}\%$$

Method 2: Express the fraction as an equivalent fraction with a denominator of 100. This method is easier if the denominator is a factor of 100.

EXAMPLE

$$\frac{1}{5} = \frac{n}{100} \rightarrow n = 20 \rightarrow \frac{20}{100} = 20\% \qquad \frac{3}{8} = \frac{n}{100} \rightarrow n = 37.5 \rightarrow \frac{37.5}{100} = 37.5\%$$

DIRECTIONS: Write each fraction as a percent.

1. $\frac{3}{5}$ **2.** $\frac{7}{8}$ **3.** $\frac{2}{3}$ **4.** $\frac{7}{1000}$

5. $\frac{9}{100}$ **6.** $\frac{1}{6}$ **7.** $\frac{67}{100}$ **8.** $3\frac{1}{2}$

9. $\frac{10}{11}$ **10.** $\frac{7}{9}$ **11.** $2\frac{5}{8}$ **12.** $\frac{1}{12}$

CHALLENGE: Do you agree or disagree with the following statement? All whole numbers and mixed numbers can be expressed as percents greater than 100. Explain your answer.

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Teaching Notes 2.16: Expressing Percents as Fractions

Expressing percents as fractions is relatively easy for most students. A common problem involves a fraction such as $\frac{1}{5}$ percent or a mixed number such as $33\frac{1}{3}$ percent.

1. Explain that "percent" means a rate per hundred. Thus, 3 percent means 3 out of 100 or $\frac{3}{100}$, 29 percent means 29 out of 100 or $\frac{29}{100}$, and 153 percent means 153 out of 100 or $\frac{153}{100}$ or $1\frac{53}{100}$.
2. Explain that all percents can be expressed as fractions, mixed numbers, or whole numbers.
3. Review the two methods on the worksheet for expressing percents as rational numbers and also review the examples.
   - Note that in method 1, students change the percent to a fraction and simplify. Point out that because percents are based on hundredths, the denominator of the fraction must be 100. Students should always simplify, if possible.
   - Explain that method 2 is best used for percents that do not convert directly to fractions with a denominator of 100. Emphasize that this method works best with percents that are fractions or mixed numbers.

EXTRA HELP:

All percents greater than or equal to 100 percent will be expressed as whole numbers or mixed numbers.

ANSWER KEY:

(1) $\frac{89}{100}$ **(2)** $\frac{3}{200}$ **(3)** $\frac{3}{50}$ **(4)** $\frac{1}{3}$ **(5)** $\frac{1}{6}$ **(6)** $1\frac{1}{2}$ **(7)** $\frac{3}{1000}$ **(8)** $\frac{5}{9}$

(Challenge) Yes. Answers may vary. Every increase of $11\frac{1}{9}$ percent results in an increase of $\frac{1}{9}$. $66\frac{2}{3}$ percent is equivalent to $66\frac{6}{9}$ percent which equals $\frac{6}{9}$ and satisfies the pattern.

Name _____ Date _____

WORKSHEET 2.16: EXPRESSING PERCENTS AS FRACTIONS

Percents can be expressed as fractions using the methods below:

Method 1: Write the percent as a fraction whose denominator is 100 and simplify.

EXAMPLES

$$15\% = \frac{15}{100} = \frac{3}{20} \qquad 25\% = \frac{25}{100} = \frac{1}{4} \qquad 100\% = \frac{100}{100} = 1$$

Method 2: Multiply the percent by $\frac{1}{100}$ and simplify the product. Use this method when the percent is a fraction or a mixed number. Be sure to change any mixed number to an improper fraction before you multiply.

EXAMPLES

$$\frac{4}{5}\% = \frac{4}{5} \times \frac{1}{100} = \frac{4}{500} = \frac{1}{125} \qquad 37\frac{1}{2}\% = \frac{75}{2} \times \frac{1}{100} = \frac{75}{200} = \frac{3}{8}$$

DIRECTIONS: Write each percent as a fraction or mixed number.

1. 89%

2. $1\frac{1}{2}\%$

3. 6%

4. $33\frac{1}{3}\%$

5. $16\frac{2}{3}\%$

6. 150%

7. $\frac{3}{10}\%$

8. $55\frac{5}{9}\%$

CHALLENGE: Consider the following pattern. $11\frac{1}{9}\% = \frac{1}{9}, 22\frac{2}{9}\% = \frac{2}{9}, 33\frac{3}{9}\% = \frac{3}{9}, 44\frac{4}{9}\% = \frac{4}{9}, \ldots$ Can this pattern be used to express $66\frac{2}{3}$ as a fraction? Explain your answer.

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Teaching Notes 2.17: Solving Percent Problems

Three types of percent problems are addressed in most algebra curriculums: finding a percent of a number, finding what percent a number is of another, and finding a number when a percent of it is known. Students often confuse the methods used to solve each type of problem.

1. Explain that percent problems can be solved using a proportion.
2. Review the information and examples on the worksheet with your students. Make sure that your students understand the proportion and are able to write it correctly. Note that students may simplify a ratio prior to solving the proportion. This will not affect the answer and usually makes the computation easier. Depending on the abilities of your students, you may wish to review the steps for solving proportions.
3. Encourage your students to solve percent problems using another method if they feel the method is easier. For example, problems such as 30 percent of 80 can easily be found by multiplying 80 times 0.3. This method is often used in the lower grades.
4. Encourage your students to double-check their work by placing their answer back in the problem to see that it is reasonable. For example, 30 percent of $80 \neq 240$. Because 30 percent is a part of 80, the answer cannot be larger than 80. In this case, the decimal point was omitted. 30 percent of $80 = 24$.

EXTRA HELP:

If the percent is less than 100 percent, the "part" is less than the "whole." If the percent is greater than 100 percent, the "part" is greater than the "whole." 100 percent means the part and the whole are equal.

ANSWER KEY:

(1) 16.5 **(2)** 52 **(3)** 400 **(4)** 48% **(5)** 60% **(6)** 14.4

(Challenge) Answers may vary. Carla knew that, mathematically, "of" means "times." Because every percent can be written as a fraction, she could set up a multiplication problem that was easy to solve. She also knew that $66\frac{2}{3}$ percent can be written as a fraction equal to $\frac{2}{3}$, making this problem particularly easy to solve.

Name _____ Date _____

WORKSHEET 2.17: SOLVING PERCENT PROBLEMS

Use the proportion, percent $= \frac{\text{part}}{\text{whole}}$, to solve different kinds of percent problems. Follow these guidelines:

1. Write the percent as a fraction. If possible, simplify the fraction. If you are trying to find the percent, write n over 100.
2. Find the number to the right of the word "of" in the problem. This is the denominator of the second fraction of the proportion. It represents the "whole." If you are trying to find this number, write n as the denominator of the second fraction.
3. The numerator of the second fraction is the number in the problem that represents the "part." If you are trying to find this number, write n as the numerator of the second fraction.

EXAMPLES

What is 30% of 90?

$$\frac{30}{100} = \frac{n}{90}$$
$$\frac{3}{10} = \frac{n}{90}$$
$$n = 27$$

What percent of 80 is 40?

$$\frac{n}{100} = \frac{40}{80}$$
$$\frac{n}{100} = \frac{1}{2}$$
$$n = 50$$

60 is 40% of what number?

$$\frac{40}{100} = \frac{60}{n}$$
$$\frac{2}{5} = \frac{60}{n}$$
$$n = 150$$

DIRECTIONS: Solve each problem.

1. What is 55% of 30?
2. 2.6 is 5% of what number?
3. 60 is 15% of what number?
4. 2.4 is what percent of 5?
5. What percent of 25 is 15?
6. Find 90% of 16.

CHALLENGE: Carla found $66\frac{2}{3}\%$ of 300 by setting up the following multiplication problem: $\frac{2}{3} \times \frac{300}{1} = 200$. Was her answer correct? Explain your reasoning.

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Teaching Notes 2.18: Finding the Percent of Increase or Decrease

Although related, an increase and percent of increase are quite different, as are a decrease and percent of decrease. Students often mistake an increase or decrease for a percent of increase or decrease.

1. Explain that a percent of increase is a rate per hundred that equals the ratio of the increase to the original amount. Similarly, a percent of decrease is a rate per hundred that equals the ratio of the decrease to the original amount.
2. Review the information and example on the worksheet with your students. Explain the formulas. Be sure that your students understand that both the increase and decrease are compared to the original amount. Discuss the example and point out that the student's average increased 10 points from his original average. Although the example illustrates finding the percent of increase, point out that the percent of decrease is found in the same manner, except that the decrease (instead of increase) is placed over the original amount in the formula.
3. Depending on the abilities of your students, you may find it helpful to review the steps for solving proportions and rounding to the nearest percent.

EXTRA HELP:

The only difference in the formulas for finding the percent of increase or decrease is the numerator of the second fraction of the proportion. For finding the percent of increase, the numerator is the "amount of increase." For finding the percent of decrease, the numerator is the "amount of decrease."

ANSWER KEY:

(1) 23% decrease **(2)** 11% increase **(3)** 10% decrease **(4)** 8% decrease

(Challenge) No. Mike received about an 8 percent increase while Juan received about a 6 percent increase.

Name _____ Date _____

WORKSHEET 2.18: FINDING THE PERCENT OF INCREASE OR DECREASE

Use the proportions that follow to find the percent of increase or decrease.

Percent of Increase

$$\frac{n}{100} = \frac{\text{amount of increase}}{\text{original amount}}$$

Percent of Decrease

$$\frac{n}{100} = \frac{\text{amount of decrease}}{\text{original amount}}$$

EXAMPLE

A student had an average of 76 in math class for the first marking period. After improving his study habits, his average rose to 86 for the second marking period. Find the percent of increase. Round your answer to the nearest percent.

$$\frac{n}{100} = \frac{10}{76}$$
$$1000 = 76n$$
$$n \approx 13\%$$

DIRECTIONS: Find the percent of increase or decrease. Round your answers to the nearest percent.

1. A pair of jeans that originally sold for $59.99 was on sale for $45.99.

2. The price of a school lunch went from $1.98 to $2.20.

3. Marie ran a five-kilometer race in 32.5 minutes. The next year she ran the same race in 29.1 minutes.

4. A home that listed for $359,000 eventually sold for $329,000.

CHALLENGE: Working part-time at a local ice-cream shop, Mike earned $64 per week. Working part-time at the same shop, Juan earned $80 per week. The manager decided to give both of them a raise of $5 per week. Did they both receive the same percent of increase? Explain your answer.

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Teaching Notes 2.19: Converting from One Unit of Measurement to Another Using the Multiplication Property of One

A common mistake students make when converting from one unit of measurement to another is dividing when they should multiply or multiplying when they should divide. Understanding the multiplication property of one can reduce the incidence of this mistake.

1. Review the multiplication property of one with your students: The product of one and any number is equal to the original number.
2. Explain that to use the multiplication property of one for conversions, students will work with fractions. The number 1 can be represented as a fraction in several ways. Offer these examples: $\frac{1 \text{ foot}}{12 \text{ inches}}$, $\frac{12 \text{ inches}}{1 \text{ foot}}$, $\frac{1000 \text{ meters}}{1 \text{ kilometer}}$. Note that the numerator and denominator of each fraction represent the same value.
3. Review the example on the worksheet with your students. Point out that the "inches" in the fractions cancel each other out, leaving "yard" as the unit of measure. Note that students must understand equivalences to complete measurement conversions. If necessary, suggest they use reference tables, which are usually included in math texts.
4. Explain that there may be other ways to complete conversions. For example, converting 3 inches to feet can easily be done by writing a fraction and simplifying. Because 12 inches = 1 foot, $\frac{3}{12} = \frac{1}{4}$. Therefore, 3 inches = $\frac{1}{4}$ foot.

EXTRA HELP:

Double-check conversion facts and be sure answers are in terms of the correct unit.

ANSWER KEY:

(1) 36 inches **(2)** $1\frac{1}{4}$ feet **(3)** $\frac{1}{10}$ mile **(4)** 48 ounces **(5)** $2\frac{1}{2}$ cups **(6)** 3 quarts **(7)** $2\frac{1}{4}$ gallons **(8)** 0.064 or $\frac{8}{125}$ kilometer

(Challenge) Mariano is correct. Explanations may vary. To convert 18 inches to feet, you must divide 18 by 12 because 12 inches equal 1 foot. 18 inches equals $1\frac{1}{2}$ feet. To convert 2 yards to feet, you must multiply 2×3. Because 3 feet equal 1 yard, 2 yards equal 6 feet.

Name _____ Date _____

WORKSHEET 2.19: CONVERTING FROM ONE UNIT OF MEASUREMENT TO ANOTHER USING THE MULTIPLICATION PROPERTY OF ONE

You can convert from one unit of measurement to another by multiplying by 1 where 1 is expressed as the ratio of two equivalent measures. Follow the example and steps below:

EXAMPLE

Change 3 inches to yards.

1. Write the number you want to convert, including the units, over 1. $\frac{3 \text{ inches}}{1}$
2. Write another fraction. Place the unit that you are converting to in the numerator. Place the unit you are converting from in the denominator. $\frac{\text{yards}}{\text{inches}}$
3. Include the numbers that relate these units. $\frac{1 \text{ yard}}{36 \text{ inches}}$
4. Multiply the fraction you wrote in step 1 by the fraction in step 3.

$$\frac{3 \text{ inches}}{1} \times \frac{1 \text{ yard}}{36 \text{ inches}} = \frac{3}{36} \text{ yard} = \frac{1}{12} \text{ yard. } 3 \text{ inches} = \frac{1}{12} \text{ yard.}$$

DIRECTIONS: Convert the following measurements.

1. 3 feet to inches
2. 15 inches to feet
3. 528 feet to miles
4. 3 pounds to ounces
5. 20 ounces to cups
6. 6 pints to quarts
7. 9 quarts to gallons
8. 64 meters to kilometers

CHALLENGE: Mariano said that when you change from a smaller unit to a larger one, you divide. And when you change from a larger unit to a smaller one, you multiply. Do you agree? Explain your answer by providing examples.

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SECTION 3

Equations and Inequalities

Teaching Notes 3.1: Writing Equations

The purpose of writing equations is to show that two expressions are equal. Common mistakes students make are writing an expression in a sentence incorrectly and not making sure that the quantities on each side of the equal sign are in fact equal.

1. Explain that word phrases can be written as algebraic expressions. For example, three times a number can be written as $3n$, where n stands for a number; twice your age can be written $2a$, where a stands for your age; and four more than something can be written as $s + 4$, where s stands for something. Depending on the abilities of your students, you may find it helpful to remind them that a variable is a letter used to represent a number. You may also wish to review 1.7: "Writing Expressions."
2. Explain that in the same way that word phrases can be used to form word sentences, algebraic expressions can be used to form equations. An equation is a number sentence that shows that two expressions are equal.
3. Offer your students this hint: The word "is" often represents the equal sign in an equation. For example, a number divided by 2 is 3 can be written as $n \div 2 = 3$.
4. Review the information and example on the worksheet with your students. Note that when writing equations, students must identify a variable to represent an unknown quantity. It is helpful to select a letter that reminds them of the quantity.

EXTRA HELP:

Equations may be true or false, depending on the value of the variable.

ANSWER KEY:

Equations and variables may vary.

(1) n = a number; $n - 5 = 12$
(2) n = a number; $2n = 24$
(3) a = Marie's present age; $a + 3 = 15$
(4) n = the number of quarters; $0.25n = \$2.25$
(5) n = a number; $n - 12 = 24$
(6) a = Quinn's present age; $a - 8 = 6$
(7) n = a number; $3 + 2n = 27$
(8) c = cost of a student's ticket; $10c = \$50$

(Challenge) Answers may vary. Jason's answer was wrong because 4 is less than 5 is an inequality, not an equation. It should be written as $4 < 5$. (Jason should pay better attention in class.)

Name _____ Date _____

WORKSHEET 3.1: WRITING EQUATIONS

An equation is a number sentence that shows two expressions are equal. An expression may be a number, a variable, or several variables and numbers. To write equations follow the steps below:

1. Choose a variable for the unknown quantity. Try to choose a letter that easily represents the quantity you are trying to find. For example, use n for a number, a for age, or t for ticket.
2. Identify the phrases and values you will represent as expressions.
3. Write two expressions.
4. Use an equal sign to show that the expressions are equal.

EXAMPLE

Write an equation for 12 more than a number is 16.
$n =$ a number. The phrase and value are "12 more than a number" and "16." The equation is $12 + n = 16$.

DIRECTIONS: Write an equation for each sentence. Choose a variable and tell what it represents.

1. 5 less than a number is 12.
2. Twice a number is 24.
3. In 3 years Marie will be 15.
4. The value of the number of quarters is $2.25.
5. A number decreased by 12 is 24.
6. 8 years ago Quinn was 6.
7. 3 more than twice a number is 27.
8. 10 student tickets cost $50.

CHALLENGE: Jason does not always pay attention in class the way he should. He vaguely remembered his teacher saying that when writing equations, the word "is" usually represents the equal sign. On his test, Jason was asked to represent "4 is less than 5" algebraically. He wrote $4 = < 5$. Why was his answer wrong?

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Teaching Notes 3.2: Solving Equations by Adding or Subtracting

When solving equations, students must understand that the most important step is to isolate the variable on one side of the equation. This is done by using inverse operations such as addition and subtraction. A common mistake in solving one-step equations is adding or subtracting the wrong number from both sides of the equation.

1. Explain to your students that in order to solve an equation, they must isolate the variable on one side of the equation.
2. Review the information and examples on the worksheet with your students. Discuss the addition property of equality and the subtraction property of equality.
   - The addition property of equality states that adding the same number to both sides of the equation does not change the value of the variable.
   - The subtraction property of equality states that subtracting the same number from both sides of the equation does not change the value of the variable.

   To help students find what numbers to add or subtract, offer examples such as $-5 +$ what number $= 0$, $3 -$ what number $= 0$, and $6 +$ what number $= 0$. (The answers are 5, 3, and -6.) Emphasize that to solve these problems, students are adding the same number to or subtracting the same number from both sides of the equation. Note that in the second example students must use the definition of subtraction to rewrite the problem as addition. Depending on the abilities of your students, you may wish to review adding and subtracting integers and rational numbers.

EXTRA HELP:

You must add or subtract the same number to or from both sides of the equation.

ANSWER KEY:

(1) Subtract 6; $x = 9$ **(2)** Add 7; $x = 27$ **(3)** Subtract 4; $y = 9$ **(4)** Add 32; $z = 28$

(5) Add 3; $x = -5$ **(6)** Add 6; $n = -3$ **(7)** Add 4; $s = 0$ **(8)** Add $\frac{3}{5}$; $n = 1$

(Challenge) Mike used a shortcut and his method is correct, but he did not show all of his work. He added -3 and 2 mentally to get -1. Then he added 1 to both sides to get $x = -9$.

Name _____ Date _____

WORKSHEET 3.2: SOLVING EQUATIONS BY ADDING OR SUBTRACTING

Follow the steps below to solve an equation by adding or subtracting:

1. Isolate the variable on one side of the equation by adding the same number to both sides or subtracting the same number from both sides.
2. Find the value of the variable.

EXAMPLES

Solve each equation.

$x - 9 = -15$ To isolate x, add 9 to each side.

$+9 \quad +9$

$x = -6$

$x - (-3) = 14$ To isolate x, rewrite the problem as $x + 3 = 14$, then subtract 3 from each side.

$x + 3 = 14$

$-3 \quad -3$

$x = 11$

DIRECTIONS: Write the number you would add to or subtract from both sides of the equation. Then solve the equation.

1. $x - (-6) = 15$

2. $x + (-7) = 20$

3. $4 - (-y) = 13$

4. $-4 = z + (-32)$

5. $-3 + x = -8$

6. $n - 6 = -9$

7. $-4 = s + (-4)$

8. $\frac{2}{5} = n - \frac{3}{5}$

CHALLENGE: Sal solved the equation $x - 3 + 2 = -10$ by adding 3 to both sides, then subtracting 2 from both sides. He got the correct answer, $x = -9$. Mike solved the same equation by adding 1 to both sides. He got the same answer as Sal. What did he do? And was his method correct?

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Teaching Notes 3.3: Solving Equations by Multiplying or Dividing

When students solve equations, they must first isolate the variable on one side of the equation. A common mistake students make when using multiplication or division to isolate the variable is to multiply or divide both sides of the equation by an incorrect number.

1. Explain that to solve equations students must first isolate the variable on one side of the equation using the multiplication or division property of equality.
   - The multiplication property of equality states that multiplying both sides of the equation by the same nonzero number does not change the value of the variable.
   - The division property of equality states that dividing both sides of the equation by the same nonzero number does not change the value of the variable.
2. Review the information and examples on the worksheet with your students. Discuss the examples on the worksheet in detail. You may find it helpful to review multiplying and dividing integers and rational numbers.

EXTRA HELP:

Be sure to check that you have isolated the variable and its coefficient is 1.

ANSWER KEY:

(1) Divide by 6; $a = -7$

(2) Multiply by 9; $x = -162$

(3) Multiply by -3; $x = -36$

(4) Divide by 24; $y = -\frac{1}{3}$

(5) Multiply by -1; $n = -100$

(6) Multiply by 3; $d = 45$

(7) Multiply by $-\frac{5}{3}$; $x = 100$

(8) Multiply by -2; $a = -48$

(9) Divide by -13; $x = -2$

(10) Multiply by 6; $y = 72$

(11) Multiply by $\frac{4}{3}$; $x = 32$

(12) Divide by -2; $y = 10$

(Challenge) Both are correct. Dividing both sides by -2 is the same as multiplying both sides by $-\frac{1}{2}$.

Name _____ Date _____

WORKSHEET 3.3: SOLVING EQUATIONS BY MULTIPLYING OR DIVIDING

Follow the steps below to solve an equation by multiplying or dividing:

1. Isolate the variable on one side of the equation by multiplying or dividing both sides of the equation by the same nonzero number.
2. Find the value of the variable.

EXAMPLES

Solve each equation.

$7x = -28$ Divide each side by 7. $\longrightarrow \frac{7x}{7} = \frac{-28}{7}; x = -4$

$-\frac{3}{4}x = 15$ Multiply each side by $-\frac{4}{3}$. $\longrightarrow -\frac{4}{3}\left(-\frac{3}{4}x\right) = -\frac{4}{3} \times \frac{15}{1}; x = -20$

DIRECTIONS: Write the number you would use to multiply or divide both sides of the equation. Then solve the equation.

1. $6a = -42$

2. $\frac{x}{9} = -18$

3. $-\frac{1}{3}x = 12$

4. $24y = -8$

5. $-n = 100$

6. $15 = \frac{1}{3}d$

7. $-\frac{3}{5}x = -60$

8. $24 = -\frac{a}{2}$

9. $26 = -13x$

10. $\frac{y}{6} = 12$

11. $\frac{3}{4}x = 24$

12. $-20 = -2y$

CHALLENGE: Martin solved the equation $-2x = 20$ by dividing both sides of the equation by -2. Brianna solved the equation by multiplying both sides by $-\frac{1}{2}$. Who is correct? Explain your answer.

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Teaching Notes 3.4: Solving Two-Step Equations with the Variable on One Side

When students solve two-step equations, they must perform two operations: addition or subtraction, then multiplication or division. Common errors include performing the operations in the incorrect order or using incorrect numbers to isolate the variable.

1. Explain that solving a two-step equation requires students to write two equivalent equations. Note that an equivalent equation is an equation that has the same solution as the original equation.
2. Explain that the first step to solving a two-step equation is to use addition or subtraction to write an equivalent equation to isolate the variable and its coefficient. You may find it helpful to review 3.2: "Solving Equations by Adding or Subtracting." The next step is to use multiplication or division to write an equivalent equation to solve for the variable. If necessary, review 3.3: "Solving Equations by Multiplying or Dividing."
3. Review the information and example on the worksheet. Make sure that your students know how the numbers that were added, subtracted, multiplied, or divided were used to write the equivalent equations. Also make certain that they know how these equations were used to solve the original equation. Depending on the abilities of your students, you may find it necessary to provide more examples. Emphasize the individual steps in solving equations.

EXTRA HELP:

Substitute your answer in the original equation. If the equation is true, your solution is correct.

ANSWER KEY:

(1) $x = 8$ **(2)** $y = -2$ **(3)** $y = -11$ **(4)** $x = 3$ **(5)** $x = -3$ **(6)** $y = -5$ **(7)** $y = 18$ **(8)** $y = -20$

(Challenge) Add 13 to both sides to get $1x = 28$. This is the same as $x = 28$ according to the multiplication property of one.

Name _____ Date _____

WORKSHEET 3.4: SOLVING TWO-STEP EQUATIONS WITH THE VARIABLE ON ONE SIDE

To solve a two-step equation, you must first add the same number to or subtract the same number from both sides of an equation, then multiply or divide both sides by the same nonzero number. Follow the steps below:

1. Decide what number you can add to or subtract from both sides of the equation so that the variable and its coefficient are isolated on one side of the equation.
2. Divide both sides of the equation by the coefficient or multiply both sides by the reciprocal of the coefficient.
3. Check the work by substituting the value of the variable into the original equation.

EXAMPLE

Solve for x : $-\frac{3}{4}x - 1 = 14$

Step 1: Add 1 to each side.

$$-\frac{3}{4}x - 1 = 14$$
$$+1 \quad +1$$
$$-\frac{3}{4}x = 15$$

Step 2: Multiply both sides by $-\frac{4}{3}$.

$$-\frac{4}{3}\left(-\frac{3}{4}x\right) = -\frac{4}{3}\left(\frac{15}{1}\right)$$
$$x = -20$$

DIRECTIONS: Solve for the variable.

1. $4x - 8 = 24$

2. $-8y + 12 = 28$

3. $2y + 12 = -10$

4. $-7 + 4x = 5$

5. $24 + 12x = -12$

6. $-7y - 5 = 30$

7. $\frac{y}{3} - 5 = 1$

8. $20 = 16 - \frac{y}{5}$

CHALLENGE: Explain why $1x - 13 = 15$ is a one-step equation. Then solve for x.

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Teaching Notes 3.5: Solving Equations Using the Distributive Property

When solving an equation that contains parentheses, students must use the distributive property to eliminate the parentheses and isolate the variable. Many students have trouble applying the distributive property and they are then unable to solve the equation correctly.

1. Discuss the distributive property with your students. $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$. Depending on the abilities of your students, you may find it helpful to provide additional examples, such as $3(x + 5)$, which is equal to $3(x) + 3(5) = 3x + 15$, and $-5(x - 2)$, which is equal to $-5(x) + (-5)(-2) = -5x + 10$.
2. Provide the following example: $2(x + 5) = 20$. Instruct your students to first rewrite the equation using the distributive property. $2x + 10 = 20$. Next have students isolate the variable by subtracting 10 from both sides and dividing by 2. The answer is $x = 5$.
3. Review the information and examples on the worksheet with your students. Explain that in some equations they may need to combine similar terms. If necessary, explain that similar terms are terms that have the same variable or variables raised to the same power, such as x and $6x$. Remind your students to check their work. (*Note:* The examples on the worksheet do not include the checks.)

EXTRA HELP:

The number outside parentheses must be multiplied by each term inside the parentheses.

ANSWER KEY:

(1) $x = 25$ **(2)** $x = -1$ **(3)** $x = 5$ **(4)** $x = -12$ **(5)** $x = 2$ **(6)** $x = 2\frac{2}{3}$ **(7)** $x = -10$ **(8)** $x = \frac{1}{3}$

(Challenge) The error was in the first step and caused subsequent errors throughout the problem. The correct steps are $-3x - 36 = 15$; $-3x = 51$; $x = -17$.

Name _____ Date _____

WORKSHEET 3.5: SOLVING EQUATIONS USING THE DISTRIBUTIVE PROPERTY

Follow the steps below to solve equations that have parentheses:

1. Use the distributive property to eliminate parentheses. The distributive property states that $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$.
2. Simplify one or both sides of the equation by combining similar terms.
3. Write equivalent equations, remembering to add or subtract before multiplying or dividing, to isolate the variable.
4. Check the work by substituting the value of the variable into the original equation.

EXAMPLES

$3(x + 2) = 18$	$2(3x - 5) + x = 4$	$-2(x - 4) = 20$
$3x + 6 = 18$	$6x - 10 + x = 4$	$-2x + 8 = 20$
$3x = 12$	$7x - 10 = 4$	$-2x = 12$
$x = 4$	$7x = 14$	$x = -6$
	$x = 2$	

DIRECTIONS: Solve each equation.

1. $5(x - 7) = 90$

2. $4(x + 2) = 4$

3. $-3(x - 5) = 0$

4. $2(x + 3) = -18$

5. $4(x + 3) + 2x = 24$

6. $2(3x + 2) = 20$

7. $-3(x + 5) = 15$

8. $-4(3x - 2) = 4$

CHALLENGE: Find the error in the problem below. Then correct it.

$$-3(x + 12) = 15$$
$$-3x + 36 = 15$$
$$-3x = -21$$
$$x = 7$$

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Teaching Notes 3.6: Solving Equations with Variables on Both Sides

The strategy for solving an equation with variables on both sides is to add or subtract a variable expression to or from both sides of the equation so that the variable is on only one side of the equation. Common errors include selecting an incorrect variable expression and not simplifying the problem first.

1. Explain that variable expressions can be added or subtracted on both sides of an equation. If necessary, remind students that a variable expression is a variable and its numerical coefficient. For example, $2x$, $\frac{1}{3}x$, and $-4x$ are variable expressions.
2. Explain that before students can add a variable expression to or subtract a variable expression from both sides of an equation, each side of the equation must be in simplest form. An equation in simplest form has no parentheses and like terms are combined. You may wish to review 3.5: "Solving Equations Using the Distributive Property."
3. Provide this example: $3x + 12 = 7x$. Note that it is already in simplest form. Ask your students what variable expression they would subtract from each side. They should subtract $3x$ to get $12 = 4x$. Dividing both sides of the equation by 4 results in the solution $x = 3$.
4. Review the information and example on the worksheet with your students. Point out that the example problem may be solved in either of two ways. Each requires the same number of steps.

EXTRA HELP:

Be sure all parentheses are eliminated and all similar terms are combined before adding or subtracting the variable expressions to or from both sides of the equation.

ANSWER KEY:

(1) $y = 4$ **(2)** $x = -17$ **(3)** $y = -13$ **(4)** $x = -19$ **(5)** $y = -30$ **(6)** $x = -7$ **(7)** $x = 15$ **(8)** $y = 13$

(Challenge) Meg did not simplify before she subtracted $3x$. The correct equivalent equation is $3x + 6 = 4x$. She should have subtracted $3x$ from both sides to get $x = 6$.

Name _____ Date _____

WORKSHEET 3.6: SOLVING EQUATIONS WITH VARIABLES ON BOTH SIDES

To solve equations that have the same variable on both sides, follow the steps below:

1. Eliminate the parentheses and simplify each side.
2. Add or subtract the variable expression to or from each side of the equation so that the variable is on only one side of the equation.
3. Solve for the variable and check your solution.

EXAMPLES

$3x + 8 = 5x - 2$ can be solved in either of two ways. (Note that both sides of the equation are already simplified.)

$3x + 8 = 5x - 2$ $-3x \quad -3x$	Subtract $3x$.	$3x + 8 = 5x - 2$ $-5x \quad -5x$	Subtract $5x$.
$8 = 2x - 2$ $+2 \quad +2$	Add 2.	$-2x + 8 = -2$ $-8 \quad -8$	Subtract 8.
$10 = 2x$		$-2x = -10$	
$\frac{10}{2} = \frac{2x}{2}$	Divide by 2.	$\frac{-2x}{-2} = \frac{-10}{-2}$	Divide by -2.
$5 = x$		$x = 5$	

DIRECTIONS: Solve each equation.

1. $9y = 5y + 16$

2. $12x + 85 = 7x$

3. $-8y = -13y - 65$

4. $59 + x = 2 - 2x$

5. $3(y + 10) = 2y$

6. $2(2x - 1) = 6(x + 2)$

7. $-18 + x = -x + 12$

8. $3(y - 1) = 2(y + 5)$

CHALLENGE: Meg solved $3(x + 2) = 4x$ by subtracting $3x$ from both sides. She found $x = 2$. Her answer was incorrect. Explain where she made her mistake and then solve the equation.

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Teaching Notes 3.7: Solving Equations with Variables on Both Sides, Including Identities and Equations That Have No Solution

Solving equations that have variables on both sides presents problems for many students. Even if they are able to solve equations that have one solution, they are often confused by equations that are true for all real numbers or that have no solution.

1. Explain that some equations have only one solution, some are true for all real numbers, and some have no solution. Depending on the abilities of your students, you may find it helpful to review 3.6: "Solving Equations with Variables on Both Sides."
2. Review the information and examples on the worksheet with your students. Explain each step fully. Make sure that your students understand the meaning of "no solution," "true statement," and "identity." Emphasize the following points:
   - If an equation is equivalent to a true statement, then the equation is true for all real numbers.
   - If an equation is equivalent to a false statement, then the equation has no solution.

EXTRA HELP:

Double-check final equations to determine if all real numbers are a solution or if there is no solution.

ANSWER KEY:

(1) $x = 5$ **(2)** No solution **(3)** Identity **(4)** No solution **(5)** $y = 0$ **(6)** No solution
(7) Identity **(8)** No solution

(Challenge) Leah's answer is incorrect. She ignored the parentheses and treated $4(x + 2)$ as $4x + 2$. She subtracted $3x$ and then 2 from both sides of the equation. The correct answer is $x = -5$.

Name _____ Date _____

WORKSHEET 3.7: SOLVING EQUATIONS WITH VARIABLES ON BOTH SIDES, INCLUDING IDENTITIES AND EQUATIONS THAT HAVE NO SOLUTION

When you solve an equation that has a variable on both sides, you may

- Find one solution.
- Obtain a false statement, which means there is no solution.
- Obtain a true statement, which means that the equation is an identity and is true for all real numbers.

EXAMPLES

$3(x-4)=x+6$	$4x+9=1+4(x+2)$	$3x-7=7+3x$
$3x-12=x+6$	$4x+9=1+4x+8$	$-7=7$
$2x-12=6$	$4x+9=9+4x$	False statement, no solution
$2x=18$	$9=9$	
$x=9$	True statement, an identity	
One solution		

DIRECTIONS: Solve each equation, if possible, or state if the equation is an identity or if it has no solution.

1. $4x+1=7(x-2)$

2. $3x+x=4x-3$

3. $2(y+3)=2y+6$

4. $5x-7=5x+9$

5. $y+9=2y+9$

6. $2y-12=2y$

7. $x+6=\frac{1}{2}(12+2x)$

8. $-\frac{1}{3}(12+9y)=4-3y$

CHALLENGE: Leah solved the equation $3x+3=4(x+2)$. She found the solution to be $x=1$. Was she correct? If her answer was wrong, explain why and find the correct answer.

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Teaching Notes 3.8: Solving Absolute Value Equations

To solve absolute value equations, students must first isolate the absolute value expression. They then must take into account that the expression within the absolute value symbols can be positive or negative, a fact that affects the solution of the equation. Errors are often made in both of these procedures.

1. Explain to your students that absolute value is the number of units a number is from 0 on the number line. Present examples such as $|3| = 3$ and $|-5| = 5$. Note that the absolute value of a number is always positive, except $|0|$, which equals 0.
2. Provide examples of absolute value expressions such as $|x - 4|$, $|3x + 2|$, and $|-2x - 10|$. The absolute value of these expressions cannot be negative even though the expressions inside the absolute value symbols may be positive or negative. For example, if $|x - 4| = 5$, $x - 4$ could equal 5 or $x - 4$ could equal -5.
3. Review the information and example on the worksheet with your students. Point out that to isolate the absolute value expression in an equation, they should follow the same procedures as for isolating a variable. They may need to add the same number to or subtract the same number from both sides of the equation, or they may need to multiply or divide both sides of the equation by the same nonzero number. Make sure that your students understand the steps for solving equations. Note that there are two solutions to the absolute value equation. Check each solution with your students.

EXTRA HELP:

Be sure to write two equations and solve each of them.

ANSWER KEY:

(1) $x = 8$ and $x = -8$ **(2)** $y = 12$ and $y = -6$ **(3)** $y = 8$ and $y = -7$ **(4)** $x = 4$ and $x = -7$
(5) $y = -3$ and $y = 7$ **(6)** $x = -3$ and $x = 3$ **(7)** $x = 9$ and $x = -3$ **(8)** $x = 2$ and $x = 8$

(Challenge) April forgot to isolate the absolute value expression. Because the equation was equal to zero, she concluded that there was only one solution. She should have found that $|3x| = 3$ and that $x = 1$ and $x = -1$.

Name _____ Date _____

WORKSHEET 3.8: SOLVING ABSOLUTE VALUE EQUATIONS

Follow the steps below to solve an absolute value equation:

1. Isolate the absolute value expression.
2. Write two equations.
   - One equation states that the expression in the absolute value symbols is positive.
   - The other equation states that the expression in the absolute value symbols is negative.
3. Solve each equation.
4. Check each solution in the original equation.

EXAMPLE

Solve for x.

$|2x + 4| - 8 = 12$
$+8 \quad +8$ Add 8 to isolate the absolute value expression.

$|2x + 4| = 20$ $2x + 4$ is written inside the absolute value symbols. It may equal 20 or -20. Write and solve two equations: $2x + 4 = 20$, $x = 8$, and $2x + 4 = -20$, $x = -12$.

DIRECTIONS: Solve each equation.

1. $|x| = 8$

2. $|y - 3| = 9$

3. $|2y - 1| = 15$

4. $|2x + 3| = 11$

5. $|-2y + 4| = 10$

6. $|2x| - 4 = 2$

7. $|x - 3| + 2 = 8$

8. $|5 - x| + 9 = 12$

CHALLENGE: April tried to solve $|3x| - 3 = 0$ and concluded that $x = 1$. Her answer was incorrect. What was her mistake? What is the correct answer?

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Teaching Notes 3.9: Solving Absolute Value Equations That Have Two Solutions, One Solution, or No Solution

Many of your students may be proficient at solving absolute value equations. However, many of these same students may become confused if, after they have isolated the absolute value expression, they find that it is equal to a negative number or that it equals zero.

1. Explain that absolute value equations may have two solutions, one solution, or no solution. Depending on the abilities of your students, you may find it helpful to review 3.8: "Solving Absolute Value Equations," which focuses on absolute value equations that have two solutions.
2. Ask your students to consider this example: $|3x + 1| = -5$. Note that the absolute value expression is isolated and that it equals -5. Because the absolute value of any number or expression must be greater than or equal to zero, this equation has no solution. Emphasize that the absolute value expression must be isolated before determining whether or not there is a solution. For example, $|3x + 1| - 7 = -5$ may appear to have no solution. Yet when it is rewritten as $|3x + 1| = 2$, the absolute value expression is isolated and there are two solutions.
3. Now ask your students to consider this example: $|2x + 4| = 0$. Because zero is neither positive nor negative, only one equation can be written: $2x + 4 = 0$, therefore $x = -2$. There is only one solution.
4. Review the information and examples on the worksheet with your students. Make sure that your students understand the steps for solving equations. If necessary, provide additional examples for review.

EXTRA HELP:

Isolate the absolute value expression before you determine the number of solutions.

ANSWER KEY:

(1) $x = 18$ and $x = -12$ **(2)** $x = -5$ **(3)** No solution **(4)** $y = 0$
(5) $x = 4$ and $x = -6$ **(6)** $y = 2$ and $y = -5$ **(7)** $x = 7$ and $x = 3$ **(8)** No solution
(9) $x = 3$ and $x = -1$ **(10)** No solution **(11)** No solution **(12)** $x = -2$ and $x = -14$

(Challenge) Equations may vary. One correct response is $|x - 2| = 0$.

Name _____ Date _____

WORKSHEET 3.9: SOLVING ABSOLUTE VALUE EQUATIONS THAT HAVE TWO SOLUTIONS, ONE SOLUTION, OR NO SOLUTION

Follow the guidelines below to solve absolute value equations:

1. Isolate the absolute value expression.
2. If the absolute value equation equals
   - A positive number, there are two solutions
   - A negative number, the equation has no solution
   - Zero, there is one solution

EXAMPLES

Positive number

$|x + 1| = 10$

Write two equations.

$x + 1 = 10 \quad x + 1 = -10$

$x = 9$ and $x = -11$

Negative number

$|2x| = -2$

No solution.

Zero

$|x - 3| = 0$

Write one equation.

$x - 3 = 0$

$x = 3$

DIRECTIONS: Solve each equation, if possible, or state if the equation has no solution.

1. $|x - 3| = 15$
2. $|x + 5| = 0$
3. $|y - 2| = -3$
4. $|y| - 3 = -3$
5. $-2|x + 1| = -10$
6. $|2y + 3| = 7$
7. $-3 + |x - 5| = -1$
8. $2|x + 7| = -1$
9. $2|x - 1| = 4$
10. $|x - 5| + 12 = 10$
11. $-3|x + 7| = 30$
12. $-4|x + 8| + 13 = -11$

CHALLENGE: Write an absolute value equation that has one solution, $x = 2$.

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Teaching Notes 3.10: Classifying Inequalities as True or False

To determine if an equality is true or false, students must first compare quantities. Combined inequalities and use of the symbols $\leq$ and $\geq$ can be particularly confusing.

1. Provide an example of a number line for your students. Review the number line's properties:
   - A number is greater than any number to its left.
   - A number is less than any number to its right.
2. Review the information and examples on the worksheet with your students. Discuss the meaning of the inequality symbols. Particularly emphasize the meanings of the $\leq$ and $\geq$ signs. Note how some inequalities can be combined. Encourage your students to sketch a number line to help them recognize the relationship between the numbers.

EXTRA HELP:

Read the inequality. Use words in place of the symbol to clarify your thinking.

ANSWER KEY:

Answers may vary. One correct answer for each false inequality follows.

(1) F; $3 < 4$ **(2)** F; $3 \leq 4$ **(3)** T **(4)** T
(5) T **(6)** F; $|-4| \leq 4$ **(7)** T **(8)** F; $(-3)^2 < 10$
(9) T **(10)** T **(11)** F; $-5 < -2^2 < 4$ **(12)** T

(Challenge) Joe is incorrect. He did not understand the meaning of the $\leq$ symbol. He seemed to interpret it as $-2 < 0$ *and* $-2 = 0$ instead of $-2 < 0$ *or* $-2 = 0$, which is a true statement.

Name _____ Date _____

WORKSHEET 3.10: CLASSIFYING INEQUALITIES AS TRUE OR FALSE

An inequality is a statement that uses an inequality symbol to compare two or more quantities. Inequality symbols follow:

$>$	is greater than	$3 > 0$
$<$	is less than	$6 < 10$
$\geq$	is greater than or equal to	$5 \geq 3$
$\leq$	is less than or equal to	$10 \leq 10$

Some inequalities may be combined:

Example: $3 < 4 < 10$ means that 4 is greater than 3 and 4 is less than 10. This is the same as saying 4 is between 3 and 10.

Example: $7 > 6 > 4$ means that 6 is greater than 4 and 6 is less than 7. This is the same as saying 6 is between 7 and 4.

DIRECTIONS: Decide if each inequality is true or false. If it is false, change one number in the inequality to make it true.

1. $3 < -4$

2. $3 \leq -4$

3. $-3 < -2 < -1$

4. $6 \geq 6$

5. $3 \geq 2 \geq 1$

6. $|-4| \leq -4$

7. $0.5 \leq 1 < 2$

8. $(-3)^2 < 9$

9. $-3^2 < 9$

10. $-2 < -1 \leq 2$

11. $-4 < -2^2 < 4$

12. $1 > \left(\frac{1}{3}\right)^2 > 0$

CHALLENGE: Joe said that $-2 \leq 0$ is false because -2 cannot be equal to zero and less than zero. Is he correct? Explain your answer.

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Teaching Notes 3.11: Writing Inequalities

Writing inequalities involves using the same skills as writing equations, except that students must translate key words that transform a comparison into a number sentence. This often proves to be a problem for some students.

1. Explain to your students that just as they used expressions to write equations, they will use expressions to write inequalities. Depending on their abilities, you may wish to review 1.7: "Writing Expressions."
2. Explain that equations indicate that two expressions are equal. Inequalities indicate that two expressions are unequal. An expression may be greater than, greater than or equal to, less than, or less than or equal to another expression.
3. Review the information and the examples on the worksheet with your students. Explain that inequality symbols are used to compare two quantities or expressions and that inequality symbols can be used to complete a number sentence when the quantities on each side of the number sentence are not equal.
4. Provide real-life situations where the inequality symbols are used. For example, a bus can hold at most 48 people. If n represents the number of people planning to take a bus, $n \leq 48$ models the situation. Likewise, if at least 20 students are needed to form a club, $n \geq 20$ models this situation.

EXTRA HELP:

Inequalities are number sentences and always contain an inequality symbol.

ANSWER KEY:

(1) $5 < 10$ **(2)** $-2 > -4$ **(3)** $3 \geq 0$ **(4)** $n - 2 \leq 16$ **(5)** $4n \geq 15$
(6) $n + (-2) \geq 5$ **(7)** $\frac{n}{5} \leq 21$ **(8)** $5n > 90$ **(9)** $n >$ 90 percent **(10)** $n \geq 300$

(Challenge) "7 less than a number" is an expression that can be written as $n - 7$. "7 is less than a number" is an inequality that can be written as $7 < n$.

Name _____ Date _____

WORKSHEET 3.11: WRITING INEQUALITIES

The symbols below are used to write inequalities. The inequality symbols show that two expressions are not equal. n represents a number.

- The symbol $<$ means "is less than." *Example:* The sum of a number and 4 is less than 10. $n + 4 < 10$
- The symbol $>$ means "is greater than." *Example:* The product of 3 and a number is greater than 15. $3n > 15$
- The symbol $\leq$ means "is less than or equal to," "is no greater than," or "is at most." *Example:* -3 times a number is at most 1. $-3n \leq 1$
- The symbol $\geq$ means "is greater than or equal to," "is at least," or "is no less than." *Example:* Twice a number is at least 8. $2n \geq 8$

DIRECTIONS: Write an inequality for each statement. Use n to represent a number in problems 4 through 10.

1. 5 is less than 10.

2. -2 is greater than -4.

3. 3 is greater than or equal to 0.

4. 2 less than a number is at most 16.

5. 4 times a number is at least 15.

6. The sum of a number and -2 is no less than 5.

7. A number divided by 5 is at most 21.

8. The product of 5 and a number is greater than 90.

9. Sam's test score exceeded 90 percent.

10. At least 300 people ran a five-kilometer race.

CHALLENGE: Explain the difference in the meaning of "7 less than a number" and "7 is less than a number."

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Teaching Notes 3.12: Solving Inequalities with Variables on One Side

The most common error students make in solving inequalities with a variable on one side is to forget to change the direction of the inequality symbol when they multiply or divide both sides of the equation by a negative number. When students understand why they need to change the direction of the inequality symbol, they are less likely to make this mistake.

1. Discuss the ways students can write equivalent inequalities:
   - Add the same number to or subtract the same number from both sides of the inequality. *Example:* $3 < 4$ can be rewritten as $3 + 5 < 4 + 5$ or $3 - 1 < 4 - 1$.
   - Multiply or divide both sides of the inequality by the same positive number. *Example:* $3 < 4$ can be rewritten as $3 \times 5 < 4 \times 5$ or $15 < 20$.

   In each case, the inequalities are equivalent.
2. Ask your students to consider what happens if you multiply both sides of an inequality by a negative number. Offer this example: $3 < 4$ but $3 \times (-2) > 4 \times (-2)$. Emphasize that the direction of the inequality symbol changed. Instruct your students to sketch a number line to recognize this relationship. Explain that the direction of an inequality symbol would also change if they were to divide both sides of an inequality by a negative number. Provide more examples if necessary.
3. Reinforce the multiplication axiom of order:
   - If $a < b$, then $ac < bc$ and if $a > b$, then $ac > bc$, where $c > 0$.
   - If $a < b$, then $ac > bc$ and if $a > b$, then $ac < bc$, where $c < 0$.
4. Review the information and the examples on the worksheet with your students.

EXTRA HELP:

Multiplying or dividing both sides of an inequality by zero will never produce an equivalent inequality.

ANSWER KEY:

(1) $x > 7$ **(2)** $y \leq 2$ **(3)** $x \leq -5$ **(4)** $x > -8$ **(5)** $y > 9$ **(6)** $x \leq 4$ **(7)** $y > 10$ **(8)** $x < 100$

(Challenge) Yes, he is correct. The variable may be placed on the left side of the inequality. $-2 > x$ is the same as $x < -2$.

Name _____ Date _____

WORKSHEET 3.12: SOLVING INEQUALITIES WITH VARIABLES ON ONE SIDE

To solve an inequality with a variable on one side, follow the steps below:

1. Add the same number to or subtract the same number from both sides to isolate the variable.
2. Multiply or divide both sides of the inequality by the same nonzero number so that the coefficient of the variable is 1.
3. When you multiply or divide both sides of the inequality by the same negative number, switch the direction of the inequality symbol.

EXAMPLES

Solve each inequality.

$$\begin{array}{r} 3x+4> \ 10 \\ -4 \quad -4 \\ \hline 3x>6 \\ \frac{3x}{3}>\frac{6}{3} \\ x>2 \end{array} \qquad \begin{array}{r} -2x-8>10 \\ +8 \quad +8 \\ \hline -2x>18 \\ \frac{-2x}{-2}<\frac{18}{-2} \\ x<-9 \end{array} \qquad \begin{array}{r} 2+5x \le -13 \\ -2 \qquad -2 \\ \hline 5x \le -15 \\ \frac{5x}{5} \le \frac{-15}{5} \\ x \le -3 \end{array}$$

DIRECTIONS: Solve each inequality.

1. $x-4>3$

2. $y+10 \le 12$

3. $-6x \ge 30$

4. $-x<8$

5. $7<2y-11$

6. $14-6x \ge -10$

7. $\frac{y}{2}-1>4$

8. $6-\frac{x}{5}>-14$

CHALLENGE: Tim solved the inequality $4<-3x-2$. He found that $6<-3x$ and that $-2>x$. The correct answer on his teacher's answer key was $x<-2$. Was Tim also correct? Explain your answer.

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Teaching Notes 3.13: Rewriting Combined Inequalities as One Inequality

When students solve combined inequalities, they must express their solution as one inequality. Many, however, have difficulty combining two inequalities as one.

1. Explain that an inequality is a number sentence that states two quantities are not equal. Inequalities may be combined by the word "or" or by the word "and."
2. Review the information and examples on the worksheet with your students. Make sure they understand that disjunctions are number sentences joined by "or" and conjunctions are number sentences joined by "and."
   - Discuss the examples of the disjunctions and point out that some, such as the third example, cannot be expressed as one inequality.
   - Discuss the examples of the conjunctions. Explain that in the first two examples, x is between two numbers. x is larger than the number on its left (in the combined inequality) and smaller than the number on its right (in the combined inequality). Note that this is the preferred way of expressing conjunctions. Also note that if a number is between two numbers, it excludes the two numbers it is between.
   - Focus the attention of your students on the last example, $-1 \geq x > -3$. This should be rewritten as $-3 < x \leq -1$, the preferred way of writing the expression. Be sure that your students understand how the inequality was rewritten: $-1 \geq x$ and $x > -3$ means the same as $-3 < x$ and $x \leq -1$. Therefore, the original inequality can be rewritten as $-3 < x \leq -1$.

EXTRA HELP:

The combined inequality $a < n < b$ is a conjunction that means n is between the numbers a and b.

ANSWER KEY:

(1) $3 < y < 7$ **(2)** $-1 \leq x$ **(3)** $-4 < x \leq 0$ **(4)** $2 < x < 5$ **(5)** $-1 < y < 1$
(6) $-10 < y \leq -3$ **(7)** $-1 \leq x < 4$ **(8)** Impossible

(Challenge) All real numbers are described because all real numbers are either less than zero or greater than or equal to zero.

Name _____ Date _____

WORKSHEET 3.13: REWRITING COMBINED INEQUALITIES AS ONE INEQUALITY

There are two types of combined inequalities: disjunctions and conjunctions. Disjunctions are two number sentences joined by "or." Conjunctions are two number sentences joined by "and."

EXAMPLES OF DISJUNCTIONS

- $x > 4$ or $x = 4$, which is the same as $x \geq 4$. This statement includes 4 or all real numbers that are greater than 4.
- $x < -1$ or $x = -1$, which is the same as $x \leq -1$. This statement includes -1 or all real numbers that are less than -1.
- $x < 3$ or $x > 4$. This statement cannot be expressed as one inequality.

EXAMPLES OF CONJUNCTIONS

- $3 < x$ and $x < 7$, which is the same as $3 < x < 7$. This statement includes all real numbers that are between 3 and 7. It does not include 3 or 7.
- $-4 \leq x$ and $x < 0$, which is the same as $-4 \leq x < 0$. This statement includes -4 and all numbers between -4 and 0. It does not include 0.
- $-1 \geq x$ and $x > -3$, which is the same as $-1 \geq x > -3$. This statement should be written as $-3 < x \leq -1$. It includes -1 and all the real numbers between -3 and -1. It does not include -3.

DIRECTIONS: Write a combined inequality for each of the following. If a statement cannot be expressed as one inequality, write "impossible."

1. $3 < y$ and $y < 7$

2. $-1 < x$ or $x = -1$

3. $-4 < x$ and $x \leq 0$

4. $x > 2$ and $x < 5$

5. $1 > y$ and $y > -1$

6. $-3 \geq y$ and $y > -10$

7. $x < 4$ and $x \geq -1$

8. $y > 3$ or $y < 1$

CHALLENGE: What numbers are described by $x < 0$ or $x \geq 0$? Explain your answer.

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Teaching Notes 3.14: Solving Combined Inequalities—Conjunctions

Solving conjunctions involves rewriting a combined inequality as two separate inequalities, solving each inequality, and sometimes rewriting the solution. Students may have trouble applying one or more of these skills.

1. Explain that a conjunction is a compound mathematical sentence that joins two sentences with the word "and." When students solve a conjunction, they find values that make both statements true.
2. Review the steps for solving conjunctions and the examples on the worksheet with your students.
   - Discuss the first example on the worksheet. Explain that this statement can be rewritten as a conjunction. Be sure that your students can rewrite the statement correctly and solve each inequality. Depending on the abilities of your students, you may find it helpful to review 3.12: "Solving Inequalities with Variables on One Side." Note the way the solution is rewritten as a combined inequality: $-3 < x < 3$. If necessary, discuss the example step by step.
   - Discuss the second example on the worksheet, noting that this statement, too, was rewritten. Explain that the direction of the inequality symbols must be reversed because both sides of the inequalities are divided by -2. Also explain the way the solution was rewritten in the preferred form. If necessary, review 3.13: "Rewriting Combined Inequalities as One Inequality."

EXTRA HELP:

Solutions of conjunctions must make both inequalities true.

ANSWER KEY:

(1) $1 < x < 11$ **(2)** $8 < y \leq 9$ **(3)** $-9 \leq y < 1$ **(4)** $1 \leq y < 5$ **(5)** $-3 < x \leq 1$
(6) $-3 < y \leq 4$ **(7)** $-1 \leq y < 1$ **(8)** $5 < x < 9$

(Challenge) Serena divided both sides of the inequality by -2 and did not change the direction of the inequality symbol. The correct answer is $-2 < y < 0$.

Name _____ Date _____

WORKSHEET 3.14: SOLVING COMBINED INEQUALITIES—CONJUNCTIONS

Two inequalities joined by the word "and" are called a conjunction. To solve a conjunction follow the steps below:

1. Rewrite each combined inequality as two inequalities joined by the word "and." The entire expression between the inequality symbols is written twice. It is greater than, or greater than or equal to, the value on the left, and less than, or less than or equal to, the value on the right.
2. Solve each inequality. Remember to change the direction of the inequality symbol if you are multiplying or dividing both sides of the inequality by a negative number.
3. Rewrite the solution so that the value of the variable is greater than one number *and* less than the other.

EXAMPLES

$-1 < x + 2 < 5$
$-1 < x + 2$ and $x + 2 < 5$
$-3 < x$ and $x < 3$
$-3 < x < 3$

$-4 < -2x \leq 10$
$-4 < -2x$ and $-2x \leq 10$
$2 > x$ and $x \geq -5$
$-5 \leq x < 2$

DIRECTIONS: Solve each conjunction.

1. $-3 < x - 4 < 7$

2. $4 < y - 4 \leq 5$

3. $-6 \leq 3 + y < 4$

4. $2 \leq 2y < 10$

5. $-1 \leq -x < 3$

6. $-3 \leq 1 - y < 4$

7. $-3 \leq 2y - 1 < 1$

8. $8 < -2 + 2x < 16$

CHALLENGE: Serena solved $0 < -2y < 4$ and found the solution to be $0 < y < -2$. This is impossible because a number cannot be less than -2 and greater than zero. Where did she make her mistake? Find the correct solution.

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Teaching Notes 3.15: Solving Combined Inequalities — Disjunctions

The first step to solving a disjunction is to write two separate inequalities. The solution of the disjunction is a solution to at least one of the inequalities (meaning one or the other or both are true). Students sometimes confuse this process with that of solving a conjunction, whose solution is a solution of both of the inequalities.

1. Explain that a disjunction is a special type of combined inequality in which two inequalities are combined by the word "or." Note that this is different from a conjunction in which two inequalities are combined by the word "and."
2. Note that disjunctions can be written as two inequalities. Each inequality must be solved independently.
3. Review the information and example on the worksheet with your students. Emphasize that students must solve both inequalities. Make sure that they understand the steps for solving inequalities. Encourage your students to sketch a number line to help them recognize the relationship.
4. Remind your students that they must change the direction of the inequality symbol if they divide both sides of an inequality by a negative number. Depending on your students' abilities, you may find it helpful to review 3.12: "Solving Inequalities with Variables on One Side." Encourage your students to check their solutions.

EXTRA HELP:

The solution of a disjunction will never be expressed with the word "and."

ANSWER KEY:

(1) $y \leq -3$ or $y \geq 1$
(2) $x < 2$ or $x > 14$
(3) $y \leq -9$ or $y \geq -1$
(4) $x < 4$ or $x > 8$
(5) $x < -5$ or $x > 4$
(6) $x \leq 1$ or $x > 5$
(7) $x > 9$ or $x < -3$
(8) $x < 5$ or $x > 7$
(9) $x < -3$ or $x > 5$

(Challenge) Eva is correct. $x \neq 4$ is another way of writing the solution $x > 4$ or $x < 4$.

Name _____ Date _____

WORKSHEET 3.15: SOLVING COMBINED INEQUALITIES — DISJUNCTIONS

Two inequalities combined by the word "or" are called a disjunction. To solve a disjunction, follow the steps below:

1. Write two separate inequalities.
2. Solve each inequality.
3. The solution of the disjunction is the solution to either of the inequalities.
4. Check the solution by substituting the value in the original inequality. If the solution is correct, it is a solution to either one of the inequalities.

EXAMPLE

Solve $3x - 2 > 7$ or $-2x - 8 > 2$.

$$\begin{array}{rcr} 3x - 2 > 7 & \text{or} & -2x - 8 > 2 \\ 3x > 9 & \text{or} & -2x > 10 \\ x > 3 & \text{or} & x < -5 \end{array}$$

The solution is all real numbers that are greater than 3 or less than -5.

DIRECTIONS: Solve each disjunction.

1. $y + 2 \leq -1$ or $y + 2 \geq 3$
2. $x + 1 < 3$ or $x - 4 > 10$
3. $y + 5 \leq -4$ or $y + 5 \geq 4$
4. $2x < 8$ or $2x > 16$
5. $-2x > 10$ or $4x > 16$
6. $3x + 1 \leq 4$ or $2x - 3 > 7$
7. $x - 3 > 6$ or $x + 12 < 9$
8. $3x < 15$ or $14 < 2x$
9. $-3x + 1 > 10$ or $2x + 1 > 11$

CHALLENGE: Eva's solution to the disjunction $15 > 4x - 1$ or $1 < 4x - 15$ is $x \neq 4$. Is she correct? Explain your answer. If she is incorrect, provide the correct answer.

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Teaching Notes 3.16: Solving Absolute Value Inequalities

An absolute value inequality may be equivalent to a conjunction or a disjunction, depending on how the absolute value compares to a positive number. Many students do not realize this and consequently have trouble solving these inequalities.

1. Explain that the first steps to solving an absolute value inequality are to isolate the absolute value expression and then write a correct inequality.
2. Explain that an absolute value inequality may be equivalent to a conjunction or a disjunction. If the expression within the absolute value symbols is $<$ or $\leq$ a positive number, it is equivalent to a conjunction. If the expression within the absolute value symbols is $>$ or $\geq$ a positive number, it is equivalent to a disjunction.
3. To show why some inequalities are equivalent to a conjunction, present this example: $|x| < 7$. Ask your students to substitute 3, -1, 8, 6, -6, and -7 for x to see if the inequality is true. They should find that 3, -1, 6, and -6 will make the statement true and conclude that $x < 7$ and $x > -7$, which can be rewritten as $-7 < x < 7$.
4. To show why some inequalities are equivalent to a disjunction, present this example: $|x| > 7$. Ask your students to substitute 8, 6, -8, 3, 15, and -5 for x to see if the inequality is true. They should find that 8, -8, and 15 will make the statement true and conclude that $x > 7$ or $x < -7$, which is a disjunction.
5. Review the information and examples on the worksheet with your students. Make sure that your students understand the steps for rewriting and solving the equivalent inequalities.

EXTRA HELP:

Absolute value inequalities will always be $>, \geq, <$, or $\leq$ a positive number.

ANSWER KEY:

(1) D; $y > 1$ or $y < -13$
(2) C; $x < 1$ and $x > -7$; $-7 < x < 1$
(3) C; $y \leq -4$ and $y \geq -8$; $-8 \leq y \leq -4$
(4) D; $y > 2$ or $y < -2$
(5) D; $y \geq 11$ or $y \leq -12$
(6) C; $x < 6$ and $x > -1$; $-1 < x < 6$
(7) D; $x > 48$ or $x < -48$
(8) C; $y \geq -5$ and $y \leq 15$; $-5 \leq y \leq 15$

(Challenge) Meg made her first mistake when she wrote $2x < -4$ instead of $2x > -4$. The solution should be $x < 2$ and $x > -2$, which can be rewritten as $-2 < x < 2$.

Name _____ Date _____

WORKSHEET 3.16: SOLVING ABSOLUTE VALUE INEQUALITIES

The following guidelines will help you to rewrite and solve absolute value inequalities:

- Isolate the absolute value expression.
- When the expression within the absolute value symbols is $<$ or $\leq$ a positive number, the inequality is equivalent to a conjunction. The absolute value expression is $<$ or $\leq$ the positive number *and* $>$ or $\geq$ the opposite of the positive number.
- When the expression within the absolute value symbols is $>$ or $\geq$ a positive number, the inequality is equivalent to a disjunction. The absolute value expression is $>$ or $\geq$ the positive number *or* $<$ or $\leq$ the opposite of the positive number.
- After rewriting the absolute value inequality as a conjunction or disjunction, the conjunction or disjunction can be solved.

EXAMPLES

$|x + 3| < 6$

conjunction

$x + 3 < 6$ and $x + 3 > -6$

$x < 3$ and $x > -9$

$-9 < x < 3$

$|2x| \geq 10$

disjunction

$2x \geq 10$ or $2x \leq -10$

$x \geq 5$ or $x \leq -5$

DIRECTIONS: Write C if the inequality can be rewritten as a conjunction or D if it can be rewritten as a disjunction. Then find the solution.

1. $|y + 6| > 7$

2. $|x + 3| < 4$

3. $|6 + y| \leq 2$

4. $|6y| > 12$

5. $|1 + 2y| \geq 23$

6. $|2x - 5| < 7$

7. $\left|\frac{1}{4}x\right| > 12$

8. $|5 - y| \leq 10$

CHALLENGE: Meg wrote $|2x| < 4$ as a conjunction: $2x < 4$ and $2x < -4$. Her solution was $x < 2$ and $x < -2$. Find her error(s).

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Teaching Notes 3.17: Solving Systems of Equations Using the Substitution Method

Although several methods can be used to solve a system of equations, the substitution method is among the most common. An error students often make is to substitute an incorrect expression for a variable.

1. Explain that a system of equations is a set of two or more equations that have the same variables. If there are two equations and two variables, the solution is an ordered pair of numbers.
2. Explain that solving a system of equations is similar to solving one equation; there are no new procedures. Students are essentially solving an equation and substituting the values. Depending on the abilities of your students, you may find it helpful to review the steps for solving equations.
3. Review the information and example on the worksheet with your students.
   - Note that the second equation was used to solve for x. The second equation could have been used to solve for y and also result in a correct solution.
   - Suggest to your students that, whenever possible, they solve an equation for a variable whose coefficient is 1. This makes computation easier and is why the second equation in the example was used to solve for x.
   - Emphasize that students must substitute the expression for a variable in the equation that has not been used. In the example, because the second equation was used to solve for x, the expression $5 - y$ was substituted for x in the first equation.

EXTRA HELP:

The substitution method is easy to use when the coefficient of one of the variables is 1 or -1.

ANSWER KEY:

(1) $x = 6; y = 3$ **(2)** $a = -4; b = -2$ **(3)** $a = -8; b = -16$ **(4)** $x = 1; y = -1$ **(5)** $x = 29; y = 11$

(Challenge) $y - (10 - 2y) = 5$ was incorrectly written. It should be $y - 10 + 2y = 5$. The correct solutions are $y = 5$ and $x = 0$.

Name _____ Date _____

WORKSHEET 3.17: SOLVING SYSTEMS OF EQUATIONS USING THE SUBSTITUTION METHOD

Follow the steps below to solve a system of equations:

1. Solve one equation for one of the variables.
2. Substitute this expression in the other equation and solve.
3. Substitute this number in either one of the original equations and solve for the other variable.
4. Check your answer by substituting the values in both of the original equations.

EXAMPLE

Solve the system of equations: $2x + 4y = 12$
$x + y = 5$

Solve for x: $x + y = 5 \quad x = 5 - y$.
Substitute $5 - y$ for x in the first equation and solve for y: $2(5 - y) + 4y = 12 \rightarrow y = 1$.
Substitute this value of y in either equation. If y is substituted in the second equation, $x + 1 = 5 \rightarrow x = 4$.

Check your answer.

$2x + 4y = 12 \qquad 2(4) + 4(1) = 12$
$x + \ y = 5 \qquad\quad 4 + 1 = 5$

DIRECTIONS: Solve each system of equations using the substitution method.

1. $x = 2y$
   $x + y = 9$

2. $a = -2 + b$
   $a + b = -6$

3. $a = b + 8$
   $2a = b$

4. $2x - 3y = 5$
   $x + 3y = -2$

5. $3y - x = 4$
   $x = 7 + 2y$

CHALLENGE: William tried to solve $x = 10 - 2y$ and $y - x = 5$. He substituted $10 - 2y$ for x in the second equation. He got $y - 10 - 2y = 5$ and found $y = -15$. By substituting $y = -15$ in the first equation he found $x = 40$. But the answer did not check. Find his error and the correct solution.

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Teaching Notes 3.18: Solving Systems of Equations Using the Addition-or-Subtraction Method

The addition-or-subtraction method is an efficient way to solve systems of equations, especially when the coefficients of one of the variables are opposites or are the same. Students may make careless errors when they use this method, especially when they use the subtraction method.

1. Explain that the addition-or-subtraction method is a way (the substitution method is yet another) for solving systems of equations.
2. Emphasize that this method is most useful when the coefficients of one of the variables are opposites or when the coefficients of one of the variables are the same. In such cases, students may add or subtract the equations, eliminating one of the variables. They will then have an equivalent equation with one variable that they can solve.
3. Review the information and examples on the worksheet with your students. Emphasize that students must add or subtract every term, including the numbers. Depending on your students' abilities, you may find it necessary to review the steps for solving equations.
4. Encourage your students to check their answer in both of the original equations, not the rewritten equations. (*Note:* The check is not included with the examples.)

EXTRA HELP:

If the variables are on both sides of the equations, rewrite the equations so that all of the variables are on one side.

ANSWER KEY:

(1) $a = 6; b = 0$ **(2)** $a = 4; b = 6$ **(3)** $x = 2; y = 2$ **(4)** $a = 0; b = -2$ **(5)** $a = -3; b = -1$

(Challenge) Yes. The second equation can be rewritten as $-5x + 8y = 8$. Addition can then be used to eliminate the y-terms. The solution is $x = 8$ and $y = 6$.

Name _____ Date _____

WORKSHEET 3.18: SOLVING SYSTEMS OF EQUATIONS USING THE ADDITION-OR-SUBTRACTION METHOD

To solve systems of equations with the addition-or-subtraction method, do the following:

1. Decide if you will add or subtract to obtain an equation that has only one variable. If the coefficients of one of the variables are opposite, add the equations. If the coefficients of one of the variables are the same, subtract the second equation from the first. Solve the equation.
2. Substitute the value of the variable in one of the original equations and solve.
3. Check your solution in both of the original equations.

EXAMPLES

Addition method

$3x - 2y = 13$

$4x + 2y = 8$ Add to eliminate the y-terms.

$7x = 21$ Solve for x.

$x = 3$

$3(3) - 2y = 13$

$9 - 2y = 13$

$-2y = 4$

$y = -2$

Subtraction method

$3x + 5y = 10$

$3x + \ y = 2$ Subtract to eliminate the x-terms.

$4y = 8$ Solve for y.

$y = 2$

$3x + 5(2) = 10$

$3x + 10 = 10$

$3x = 0$

$x = 0$

DIRECTIONS: Solve each system of equations.

1. $2a - 3b = 12$
$a + 3b = 6$

2. $4a - b = 10$
$-4a + 8b = 32$

3. $5x - 5y = 0$
$5x - 2y = 6$

4. $a - b = 2$
$-3a + b = -2$

5. $8a + 3b = -27$
$2a - 3b = -3$

CHALLENGE: Can $x - 8y = -40$ and $-5x = 8 - 8y$ be solved using the addition-or-subtraction method? Explain your reasoning and solve the problem, if possible.

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Teaching Notes 3.19: Solving Systems of Equations Using Multiplication with the Addition-or-Subtraction Method

Using multiplication with the addition-or-subtraction method is yet another way to solve systems of equations. A trouble spot for students is deciding what number or numbers to multiply by.

1. Explain that using multiplication with the addition-or-subtraction method to solve systems of equations involves two major steps: multiplying one or both of the equations by a nonzero number to make the coefficients of one of the variables the opposite or the same and then using the addition-or-subtraction method to solve the equations.
2. Present this example to your students: Solve $4x - 9y = 1$ and $-5x + 6y = 4$. Ask them what they would do to eliminate a term. Explain that there are two possibilities:
   - They could multiply the first equation by 2 and the second equation by 3 and add to eliminate the y-terms.
   - They could multiply the first equation by 5 and the second equation by 4 and add to eliminate the x-terms.

   Demonstrate how students may solve the system of equations using each method. The solution is $x = -2$ and $y = -1$.
3. Review the information and example on the worksheet with your students. Note that in the example the method used eliminates the x-terms. Point out that the system could also be solved by eliminating the y-terms. The solution would be the same. (*Note:* The check is not included with the example.) If necessary, review the addition-or-subtraction method with your students.

EXTRA HELP:

After you have added or subtracted, one term should be eliminated. If no terms are eliminated, check your work.

ANSWER KEY:

(1) $x = 5; y = 3$ **(2)** $x = -1; y = 1$ **(3)** $x = -2; y = -3$ **(4)** $x = 3; y = 4$ **(5)** $x = 5; y = -2$

(Challenge) It is more efficient to eliminate the x-terms first because only the first equation would be multiplied by 2. Choosing to eliminate the y-terms first would require multiplying two equations (the first by 5 and the second by 3). The solution is $x = 1$ and $y = 3$.

Name _____ Date _____

WORKSHEET 3.19: SOLVING SYSTEMS OF EQUATIONS USING MULTIPLICATION WITH THE ADDITION-OR-SUBTRACTION METHOD

Follow the steps below to solve systems of equations using multiplication with the addition-or-subtraction method:

1. Multiply one or both of the equations by a nonzero number to obtain two equations with the same or opposite coefficients of a variable.
2. Follow the steps for using the addition-or-subtraction method:
   - Add or subtract the equations to eliminate a variable.
   - Solve the new equation.
   - Substitute the value of the variable in one of the original equations and solve.
3. Check the solution by substituting both values in each of the original equations.

EXAMPLES

Solve $3x - 5y = 8$ and $-2x + 3y = 3$. (To eliminate the x-terms, multiply the first equation by 2 and the second equation by 3.)

$$\begin{array}{rrl} 3x - 5y = 8 \rightarrow & 6x - 10y = 16 & \\ -2x + 3y = 3 \rightarrow & \underline{-6x + \ 9y = 9} & \text{Add} \\ & -y = 25 & \\ & y = -25 & \text{Substitute } -25 \text{ for } y \text{ in one of the original equations} \\ & & \text{and solve. } 3x - 5(-25) = 8;\ x = -39. \end{array}$$

DIRECTIONS: Solve each system of equation by using multiplication with the addition-or-subtraction method.

1. $3x + 2y = 21$
$7x - 11y = 2$

2. $9y + 7x = 2$
$2y - 4x = 6$

3. $-2x + 4y = -8$
$3x - 6y = 12$

4. $-6x + 3y = -6$
$2x + 6y = 30$

5. $4y + 4x = 12$
$2y + x = 1$

CHALLENGE: Consider the solution of $2x + 3y = 11$ and $4x - 5y = -11$. Is it more efficient (less work) to eliminate the y-terms first? Or is it more efficient to eliminate the x-terms first? Explain your reasoning and solve the system of equations.

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Teaching Notes 3.20: Solving Systems of Equations Using a Variety of Methods

Solving systems of equations involves selecting an appropriate method for solving the system and obtaining a solution. Choosing the most efficient method can be challenging for many students.

1. Explain to your students that there are three methods they can use to solve systems of equations: (1) substitution, (2) addition or subtraction, and (3) multiplication with addition or subtraction. Depending on the system of equations, one method may be more efficient than the others. (A summary of when to use each method is included on the student worksheet.)
2. Emphasize that regardless of which method is used, students should always try to obtain an equation with one variable. They will then solve this equation and substitute the value of the variable in one of the original equations to find the value of the other variable.
3. Review the information on the worksheet with your students. Note that some systems of equations may be solved by more than one method. Remind your students to always check their work by substituting both values in the original equations.

EXTRA HELP:

Selecting the most efficient method for solving a system of equations makes the overall work easier.

ANSWER KEY:

(1) $x = 3;\ y = 0$ **(2)** $x = 6; y = 3$ **(3)** $x = 3;\ y = 1$ **(4)** $x = 4;\ y = 0$ **(5)** $x = 2;\ y = 1$
(6) $x = -3;\ y = -2$ **(7)** $x = 3;\ y = 2$ **(8)** $x = 3;\ y = 1$ **(9)** $x = 2;\ y = 0$ **(10)** $x = 45;\ y = -3$
(11) $x = -100;\ y = 72$ **(12)** $x = 8;\ y = 4$

(Challenge) This is a system of equations when the coefficient of one of the variables is 1 and the coefficient of the same variable in the other equation is -1.

Name _____ Date _____

WORKSHEET 3.20: SOLVING SYSTEMS OF EQUATIONS USING A VARIETY OF METHODS

There are three methods for solving a system of equations:

1. *Substitution* — Use this method when the coefficient of one of the variables is 1 or −1.
2. *Addition or subtraction* — Use this method when the coefficients of one of the variables are the same or are opposites.
3. *Multiplication with addition or subtraction* — Use this method when the substitution and addition-or-subtraction methods cannot be used.

DIRECTIONS: Select the best method and solve each system of equations.

1. $x + y = 3$
$-2y + x = 1$

2. $x - 2y = 0$
$x + 2y = 12$

3. $x + 2y = 5$
$-2x + 3y = -3$

4. $4x + 3y = 16$
$2x - 3y = 8$

5. $5x - 2y = 8$
$3x - 5y = 1$

6. $3x - 8y = 7$
$x + 2y = -7$

7. $2x + 3y = 12$
$3x + 2y = 13$

8. $x + 2y = 5$
$-2x + 3y = -3$

9. $2x + 3y = 4$
$5x - 3y = 10$

10. $x = -15y$
$x + 4y = 33$

11. $2x + 3y = 16$
$3x + 4y = -12$

12. $5x = 10y$
$-4x + 10y = 8$

CHALLENGE: Describe a system of equations that can be solved most efficiently by either the substitution method or the addition-or-subtraction method.

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Teaching Notes 3.21: Solving Systems of Equations That Have One Solution, No Solution, or an Infinite Number of Solutions

Solving systems of equations poses a special problem when students attempt to eliminate a variable and find that both variables are eliminated. Accustomed to finding a specific value for a variable, some students become confused as to what to do next.

1. Explain that many systems of equations have one solution, others have no solution, and still others may have an infinite number of solutions.
2. Remind your students that systems of equations may be solved using the substitution method, the addition-or-subtraction method, or the multiplication with addition-or-subtraction method. Note that they should always try to choose the most efficient method for solving a system of equations. Depending on the abilities of your students, you may wish to review 3:20: "Solving Systems of Equations Using a Variety of Methods."
3. Review the information and examples on the worksheet with your students. Emphasize that if both variables are eliminated, the system of equations may have an infinite number of solutions or it may have no solution. The answer depends on whether the equation is true or false. A true statement indicates the system has an infinite number of solutions. A false statement indicates the system has no solution.

EXTRA HELP:

Sometimes it is necessary to rewrite one or both equations to "line up the variables" when using the addition-or-subtraction method or the multiplication with addition-or-subtraction method.

ANSWER KEY:

(1) No solution **(2)** $x = 0; y = 4$ **(3)** $x = 3; y = -3$ **(4)** Infinite number of solutions **(5)** $x = -2; y = -3$

(Challenge) Jay is incorrect. If he solved the system of equations, he would find that both variables would be eliminated, leaving a true statement: $0 = 0$. This system has an infinite number of solutions. $x = 1$ and $y = 2$ is one of them.

Name _____ Date _____

WORKSHEET 3.21: SOLVING SYSTEMS OF EQUATIONS THAT HAVE ONE SOLUTION, NO SOLUTION, OR AN INFINITE NUMBER OF SOLUTIONS

To solve a system of equations, try to eliminate one of the variables and then follow the steps below:

1. If only one of the variables is eliminated, the system of equations has only one solution. You can find this solution by substituting the value for the variable in one of the equations and solving for the other variable.
2. If both variables are eliminated and the equation is true, there is an infinite number of solutions.
3. If both variables are eliminated and the equation is false, there is no solution.

EXAMPLES

Solve each system of equations.

$y - x = 1$
$y + x = 9$
$2y = 10$
$y = 5$

Substitute 5 for y.
$5 + x = 9$
$x = 4$

$-x + 2y = 2$
$x - 2y = -2$
$0 = 0$

True statement

Infinite number of solutions

$2x + y = 4$
$-2x - y = 6$
$0 = 10$

False statement

No solution

DIRECTIONS: Solve each system of equations, or write "infinite number of solutions" or "no solution."

1. $x = 2y$
$x - 2y = 4$

2. $x + 2y = 8$
$x - y = -4$

3. $x - y = 6$
$2x + y = 3$

4. $2x + 2y = 10$
$x + y = 5$

5. $-3x + 4y = -6$
$5x - 6y = 8$

CHALLENGE: By using guess and check, Jay found that the solution to the system of equations $x + y = 3$ and $2(x + y) = 6$ is $x = 1$ and $y = 2$. Do you agree? Explain your answer.

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Teaching Notes 3.22: Using Matrices—Addition, Subtraction, and Scalar Multiplication

Many students find matrices confusing. Much of the confusion is a result of the specialized vocabulary of matrices.

1. Explain that a matrix is a set of numbers arranged in rows and columns, usually enclosed in brackets. Point out that a matrix is composed of rows and columns. A matrix is named by a capital letter.
2. Explain that the number of rows and columns represent the dimensions of a matrix. For example, a matrix with three rows and two columns is a 3×2 matrix. The number of rows precedes the number of columns.
3. Explain that the numbers in a matrix are called the "elements," or "entries," of the matrix.
4. Explain the steps for adding or subtracting matrices. Matrices must have the same dimensions if they are to be added or subtracted. To add or subtract matrices, add or subtract corresponding elements to obtain a matrix of the same dimension. If matrices have different dimensions, they cannot be added or subtracted.
5. Explain that to multiply a matrix by a number, which is called a "scalar," students must multiply each element by the scalar.
6. Review the information and examples on the worksheet with your students. Remind them that each element of a matrix is an integer. They must follow the rules for adding, subtracting, and multiplying integers. They should be particularly careful when working with negative numbers.

EXTRA HELP:

If two matrices have different dimensions, they cannot be added or subtracted, but every matrix, regardless of its dimensions, can be multiplied by a scalar.

ANSWER KEY:

(1) $\begin{bmatrix} -2 & 4 \\ 2 & 9 \end{bmatrix}$ **(2)** $\begin{bmatrix} 1 & 1 \\ -1 & 9 \end{bmatrix}$ **(3)** $\begin{bmatrix} 6 & -8 \\ -12 & -16 \end{bmatrix}$ **(4)** $\begin{bmatrix} -3 & 3 \\ 3 & 0 \end{bmatrix}$ **(5)** Impossible

(6) $\begin{bmatrix} 3 & 9 \\ -9 & 24 \\ -12 & 18 \end{bmatrix}$ **(7)** $\begin{bmatrix} 0 & 3 \\ 10 & -5 \\ 8 & 3 \end{bmatrix}$ **(8)** Impossible

(Challenge) Yes, provided they have the same dimensions.

Name _____ Date _____

WORKSHEET 3.22: USING MATRICES—ADDITION, SUBTRACTION, AND SCALAR MULTIPLICATION

Matrices can be added, subtracted, or multiplied by a number called a scalar. Follow the guidelines below:

- Two matrices can be added or subtracted only if they have the same number of rows and columns. Add or subtract the corresponding elements.
- To multiply a matrix by a scalar, multiply each element by the scalar.

EXAMPLES

$$A = \begin{bmatrix} -3 & -4 & 1 \\ 1 & 6 & 7 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 7 & 6 \\ 2 & 4 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} -4 & 3 & 7 \\ 3 & 10 & 15 \end{bmatrix} \quad A - B = \begin{bmatrix} -2 & -11 & -5 \\ -1 & 2 & -1 \end{bmatrix} \quad 4A = \begin{bmatrix} -12 & -16 & 4 \\ 4 & 24 & 28 \end{bmatrix}$$

DIRECTIONS: Use these matrices to perform the indicated operations, if possible. If not possible, write "impossible."

$$C = \begin{bmatrix} -3 & 4 \\ 6 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 1 \\ 3 & 8 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 3 \\ -3 & 8 \\ -4 & 6 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 6 \\ 7 & 3 \\ 4 & 9 \end{bmatrix}$$

1. $C + D$ **2.** $D + E$ **3.** $-2C$ **4.** $C - E$

5. $N + E$ **6.** $3N$ **7.** $M - N$ **8.** $M + C$

CHALLENGE: Is it possible to add three or more matrices? Explain your reasoning.

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Teaching Notes 3.23: Identifying Conditions for Multiplying Two Matrices

Multiplying two matrices can be challenging for many students. They must be able to identify which matrices can be multiplied and they often have trouble with the concept that multiplication with matrices is not commutative.

1. Review the matrix vocabulary, especially row, column, and dimensions. Note that a row is horizontal and a column is vertical. The number of rows and columns determines the dimensions of a matrix. Thus, a 3×2 matrix has three rows and two columns.
2. Explain the notation: $A_{m \times n}$. This is a type of shorthand for saying Matrix A has m rows and n columns.
3. Review the information and example on the worksheet with your students. Point out that in the example the product $A \times B$ can be found. Ask them if $B \times A$ can be found. Explain that this is impossible because B has one column and A has two rows. Emphasize that multiplying matrices is not always commutative. In order to multiply two matrices, the number of columns in the first matrix must be the same as the number of rows in the second.
4. Carefully go over the directions for the problems with your students. Be sure that they understand they must find the dimensions of each matrix first before they multiply the matrices.

EXTRA HELP:

Do not assume that multiplication of two matrices is commutative. The commutative property applies only if the number of columns in the first matrix is the same as the number of rows in the second matrix.

ANSWER KEY:

(1) $P_{2\times2}$ **(2)** $P_{2\times2}$ **(3)** $P_{2\times1}$ **(4)** Cannot be multiplied **(5)** Cannot be multiplied **(6)** $P_{2\times1}$

(Challenge) Agree. If the number of all of the rows and all of the columns are the same, the number of columns in the first matrix must be the same as the number of rows in the second.

Name _____ Date _____

WORKSHEET 3.23: IDENTIFYING CONDITIONS FOR MULTIPLYING TWO MATRICES

To multiply two matrices, the number of columns in the first matrix must be the same as the number of rows in the second matrix. (If this is not true, the matrices cannot be multiplied.) The product of two matrices is a matrix that has the same number of rows as the first matrix and the same number of columns as the second matrix. This is summarized by the formula: $A_{m\times n} \times B_{n\times q} = P_{m\times q}$. P is a matrix that represents the product of $A \times B$. P has m rows and q columns.

EXAMPLE

Determine if P, the product of $A \times B$, can be found. If it can, find the dimensions of P. $A_{2\times 3} \times B_{3\times 1} = P_{2\times 1}$ Matrix A has three columns, which is the same as the number of rows of Matrix B, therefore the product can be found. The dimensions of P are 2×1.

DIRECTIONS: Use the matrices below to determine if the product of the matrices in each problem can be found. If the product can be found, find the dimensions of the product. If the product cannot be found, write "cannot be multiplied."

$$L = \begin{bmatrix} 3 & 4 \\ -1 & 6 \end{bmatrix} \quad M = \begin{bmatrix} -3 & 4 \\ 2 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 14 \\ 6 & -15 \\ -10 & 4 \end{bmatrix}$$

1. $L \times M$

2. $M \times L$

3. $M \times C$

4. $C \times M$

5. $C \times D$

6. $L \times C$

CHALLENGE: Samantha said that if a matrix has the same number of rows and columns, it can be multiplied by any other matrix that has an identical number of rows and columns. Do you agree with her? Explain your reasoning.

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Teaching Notes 3.24: Multiplying Two Matrices

Many students find multiplying two matrices to be a complicated process. Although the basic procedure is relatively simple, students are often uncertain of where to place each element in the product matrix.

1. Explain that only some matrices can be multiplied. Emphasize that in order to multiply two matrices, the number of columns in the first matrix must equal the number of rows in the second matrix. Only after students have determined that two matrices can be multiplied can they move ahead and multiply them. You might find it helpful to review 3.23: "Identifying Conditions for Multiplying Two Matrices" with your students. Also review, if necessary, how students can find the dimensions of the product matrix.
2. Review the information and example on the worksheet with your students. Be sure that students understand the precise steps for the multiplication of matrices. Caution them to pay especially close attention to the order for multiplying the elements. Also remind them to pay close attention to integers with negative signs.

EXTRA HELP:

Check the dimensions of the product matrix to be sure your elements are placed correctly.

ANSWER KEY:

(1) $\begin{bmatrix} -3 \\ 36 \end{bmatrix}$ **(2)** $\begin{bmatrix} 1 & 6 \\ 2 & -16 \end{bmatrix}$ **(3)** $\begin{bmatrix} 5 & 9 \\ -8 & -20 \end{bmatrix}$ **(4)** Cannot be multiplied

(5) Cannot be multiplied **(6)** $\begin{bmatrix} 14 & 28 \\ 16 & 12 \\ 5 & 2 \end{bmatrix}$

(Challenge) Answers will vary. The matrices in numbers 4 and 5 cannot be multiplied. Students should explain that the number of columns in the first matrix differs from the number of rows in the second matrix.

Name _____ Date _____

WORKSHEET 3.24: MULTIPLYING TWO MATRICES

Follow the steps below to multiply two matrices:

1. Find the dimensions of the product matrix. The dimensions of the product matrix are the number of rows of the first matrix times the number of columns in the second matrix.
2. Find the product.
   - Multiply the elements in the first row of the first matrix by the elements in the first column of the second matrix and add. Place the element in the first row of the product matrix. Continue this process until the elements in the first row of the first matrix are multiplied by the elements in each column of the second matrix.
   - Multiply the elements in the second row of the first matrix by the elements in the first column of the second matrix and add. Place this element in the second row of the product matrix. Continue this process until the elements in the second row of the first matrix are multiplied by the elements in each column of the second matrix.
   - Continue until every row is multiplied by every column.

EXAMPLE

$$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \times \begin{bmatrix} -3 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 \times (-3) + 4 \times 0 & 2 \times (-1) + 4 \times 5 \\ 1 \times (-3) + 6 \times 0 & 1 \times (-1) + 6 \times 5 \end{bmatrix} = \begin{bmatrix} -6 & 18 \\ -3 & 29 \end{bmatrix}$$

DIRECTIONS: Use the matrices below to find the product, if possible. If the matrices cannot be multiplied, write "cannot be multiplied."

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 6 \\ 8 & 4 \\ 3 & 1 \end{bmatrix}$$

1. $A \times B$ **2.** $A \times C$ **3.** $C \times A$

4. $D \times B$ **5.** $A \times E$ **6.** $E \times A$

CHALLENGE: It is impossible to multiply the matrices in some of the problems above. Select one of these problems and explain why the matrices cannot be multiplied.

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SECTION 4

Graphs of Points and Lines

Teaching Notes 4.1: Graphing on a Number Line

Many students experience confusion when graphing inequalities on a number line. Knowing the meanings of "closed" as opposed to "open" circles and the directions of arrows can eliminate confusion.

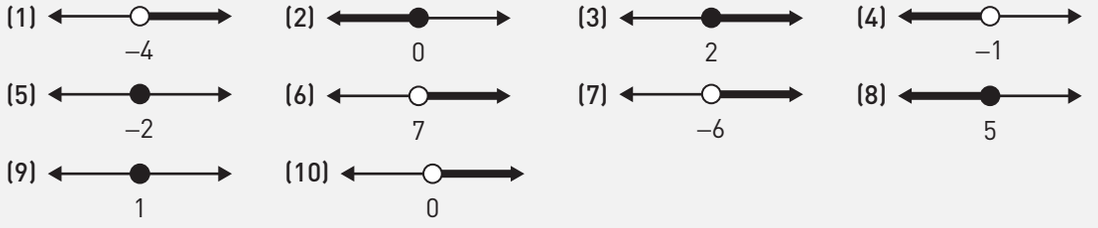
1. Present an example of a number line and review its characteristics with your students. Note that every point on the number line can be paired with a real number, which is called the "coordinate" of the point.
2. Discuss the properties of the number line. Larger numbers are located to the right of smaller numbers. Provide examples such as $5 > 4$, $3 > -1$, and $-5 > -6$. Smaller numbers are located to the left of larger numbers; for example, $4 < 5$, $-1 < 3$, and $-6 < -5$.
3. Review the information and examples on the worksheet with your students.
   - Explain that a closed circle on the number line is the graph of the number that is paired with the point; for example, $x = 3$. If a number line is shaded to the right, for example, as $x \geq 3$, the graph shows 3 and all numbers that are greater than 3, including decimals and fractions. Note that $x \leq 3$ shows 3 and all the numbers less than 3.
   - Explain that an open circle on the number line does not include the number that is paired with the point on the number line. If a number line is shaded to the right, for example $x > 3$, the graph shows all the numbers that are greater than 3. If a number line is shaded to the left, for example $x < 3$, the graph shows all the numbers less than 3.

EXTRA HELP:

Rewrite the inequalities so that the variable is on the left. Then the tip of the arrow points in the direction the number line is shaded.

ANSWER KEY:

A sketch of each graph is shown.



(Challenge) Because the inequalities are not equal to 5, the graph is a number line with an open circle on the point paired with 5. The number line is shaded to the right of 5 and to the left of 5.

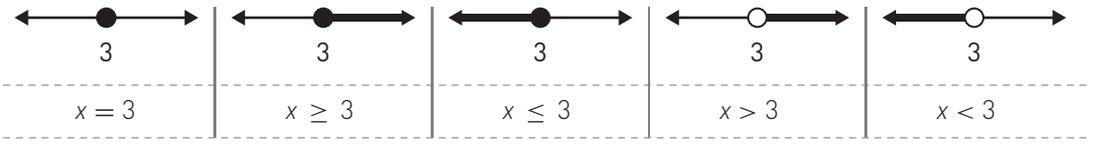
Name _____ Date _____

WORKSHEET 4.1: GRAPHING ON A NUMBER LINE

Follow the steps below to graph equations and inequalities with one variable on a number line:

1. Find the point with the given coordinate on the number line.
2. Determine the type of circle to draw.
   - Use a closed circle when x is $\geq$, $\leq$, or $=$ to a number.
   - Use an open circle when x is $>$, $<$, or $\neq$ to a number.
3. Draw an arrow in the direction of the numbers that satisfy the inequality.

EXAMPLES



DIRECTIONS: Graph each equation or inequality on a number line.

1. $x > -4$ **2.** $x \leq 0$ **3.** $x \geq 2$ **4.** $x < -1$ **5.** $x = -2$

6. $7 < x$ **7.** $-6 < x$ **8.** $5 \geq x$ **9.** $x = 1$ **10.** $0 < x$

CHALLENGE: Describe the graph of $x \neq 5$.

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Teaching Notes 4.2: Graphing Conjunctions

Graphing conjunctions is a skill that requires students to interpret inequalities and graph them according to their interpretations. Incorrect interpretations lead to faulty graphs.

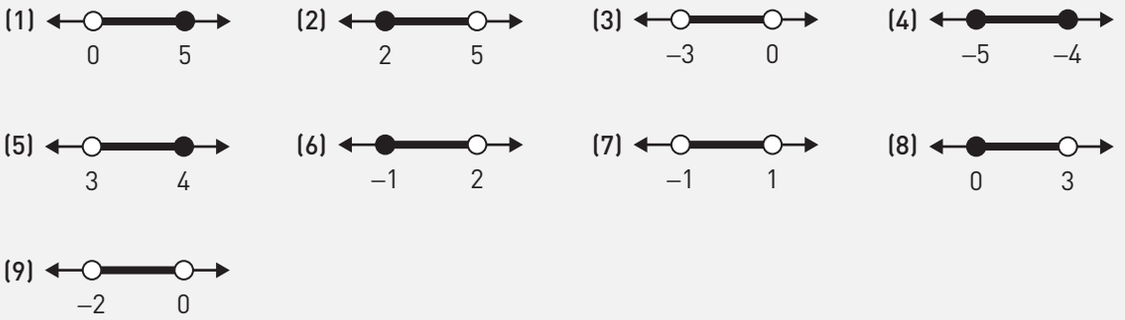
1. Explain to your students that a conjunction is a sentence formed by joining two sentences with the word "and." Two inequalities can be joined by "and" to form a conjunction.
2. Explain that all conjunctions with one variable can be graphed on a number line. Depending on the abilities of your students, you may find it helpful to present a simple number line and review its properties. Note that a conjunction is true when both sentences are true.
3. Review the information and example graphs on the worksheet with your students. Substitute the numbers -1 for a and 3 for b and discuss each graph. For example $-1 \leq x \leq 3$ contains the numbers -1 and 3 and all of the numbers between them. Remind your students that this includes decimals and fractions. Also note that the graph has closed circles. This means that the number paired with the point is included on the graph. Contrast this to the fourth graph, $a < x < b$. An open circle means that the number paired with the point is not included on the graph.
4. Review rewriting two inequalities as one. For example, $x > 0$ and $1 \geq x$ can be written as $0 < x$ and $x \leq 1$, which is the same as $0 < x \leq 1$.

EXTRA HELP:

Writing a conjunction as one combined conjunction is easier to graph.

ANSWER KEY:

Note: The number lines are not drawn to scale.



(Challenge) $-3 \leq x < 1$.

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WORKSHEET 4.2: GRAPHING CONJUNCTIONS

Two inequalities joined by the word "and" form a conjunction. A conjunction is true when both of the sentences are true. Every conjunction with one variable can be graphed on a number line.

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EXAMPLES

a and b are real numbers, $a < b$.	
$a \leq x \leq b$ includes a, b, and the numbers between them.	
$a < x \leq b$ includes b and the numbers between a and b.	
$a \leq x < b$ includes a and the numbers between a and b.	
$a < x < b$ includes the numbers between a and b.	

DIRECTIONS: Graph each conjunction on a number line.

1. $0 < x \leq 5$

2. $2 \leq x < 5$

3. $-3 < x < 0$

4. $x \leq -4$ and $x \geq -5$

5. $x > 3$ and $x \leq 4$

6. $x \geq -1$ and $x < 2$

7. $x < 1$ and $x > -1$

8. $3 > x$ and $x \geq 0$

9. $-2 < x$ and $x < 0$

CHALLENGE: Use this graph to write an inequality.

Teaching Notes 4.3: Graphing Disjunctions

Disjunctions with one variable can be graphed on a number line. Some students mistakenly believe that, like conjunctions, disjunctions may also be rewritten as one inequality.

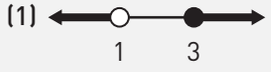
1. Explain that a disjunction is a sentence formed by joining two sentences with the word "or." Note that if at least one of the statements is true, then the disjunction is true. A disjunction cannot be rewritten as one combined inequality.
2. Explain that graphing a disjunction is similar to graphing an inequality, except that two inequalities are graphed on the same number line.
3. Review the information and example graphs on the worksheet with your students. Note that a closed circle shows that the number paired with the point is included on the graph. An open circle shows that the number paired with the point is not included on the graph.

EXTRA HELP:

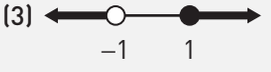
The shadings on the graph of a disjunction point in opposite directions. One part of the graph is shaded to the right and the other part is shaded to the left.

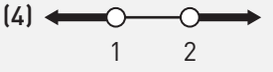
ANSWER KEY:

Note: The number lines are not drawn to scale.

(1) 

(2)

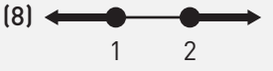
(3) 

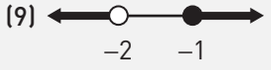
(4) 

(5)

(6)

(7)

(8) 

(9) 

(Challenge) The graph of a disjunction would model the situation. There would be an empty circle on the number line at 5 and 10. The number line would be shaded to the right of 10 and to the left of 5.

Name _____ Date _____

WORKSHEET 4.3: GRAPHING DISJUNCTIONS

Two inequalities joined by the word "or" form a disjunction. A disjunction is true when at least one of the sentences is true. Every disjunction with one variable can be graphed on a number line.

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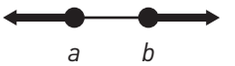
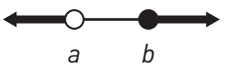
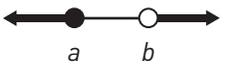
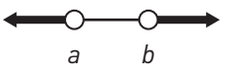
EXAMPLES

a and b are real numbers, $a < b$.	
$x \leq a$ or $x \geq b$ includes the numbers less than or equal to a or the numbers greater than or equal to b.	
$x < a$ or $x \geq b$ includes the numbers less than a or the numbers greater than or equal to b.	
$x \leq a$ or $x > b$ includes the numbers less than or equal to a or the numbers greater than b.	
$x < a$ or $x > b$ includes the numbers less than a or the numbers greater than b.	

DIRECTIONS: Graph each disjunction on a number line.

1. $x < 1$ or $x \geq 3$

2. $x \leq 0$ or $x \geq 5$

3. $x < -1$ or $x \geq 1$

4. $x < 1$ or $x > 2$

5. $-1 \geq x$ or $x \geq 3$

6. $7 \geq x$ or $x > 10$

7. $0 \geq x$ or $3 < x$

8. $x \geq 2$ or $x \leq 1$

9. $x \geq -1$ or $x < -2$

CHALLENGE: Describe the graph of the inequality that models this situation: a number larger than 10 or less than 5.

Teaching Notes 4.4: Graphing Ordered Pairs on the Coordinate Plane

Graphing points on the coordinate plane is relatively easy for students, provided they remember that they must move to the right or left before moving up or down on the graph. Some students, however, require additional practice to remember to move in the proper directions.

1. Review the coordinate plane with your students. Emphasize the following:
   - A plane is a flat surface divided into four sections called "quadrants."
   - The quadrants are formed by the intersection of a horizontal number line called the *x*-axis and a vertical number line called the *y*-axis.
   - The origin (0, 0) is the point where the *x*-axis and *y*-axis intersect.
2. Explain that the coordinate plane can be used to graph ordered pairs. The graph of an ordered pair is called a "point." For (x, y), the *x*-coordinate indicates how many units to move right or left along the *x*-axis. The *y*-coordinate indicates how many units to move up or down along the *y*-axis. All movement begins at the origin.
3. Emphasize that order matters. (x, y) differs from (y, x). For example, point (2, 5) is not the same point as point (5, 2).
4. Review the procedure for graphing ordered pairs on the worksheet with your students.
5. Provide an example showing how $(-3, 4)$ is graphed on the coordinate plane. Explain that students must start at (0, 0), then move three units to the left, four units up, and finally graph the point.

EXTRA HELP:

To graph (x, y), remember that *x* is written before *y*. Move horizontally before moving vertically.

ANSWER KEY:

(1) E	**(2)** B	**(3)** J
(4) A	**(5)** F	**(6)** I
(7) H	**(8)** C	**(9)** D

(Challenge) Yes, $(x, y) = (y, x)$ but only if the *x*-coordinate is the same as the *y*-coordinate.

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WORKSHEET 4.4: GRAPHING ORDERED PAIRS ON THE COORDINATE PLANE

Graphing points (called "ordered pairs") requires moving horizontally (across) or vertically (up or down) on the coordinate plane, depending on the values of x and y. To graph (x, y), follow the steps below:

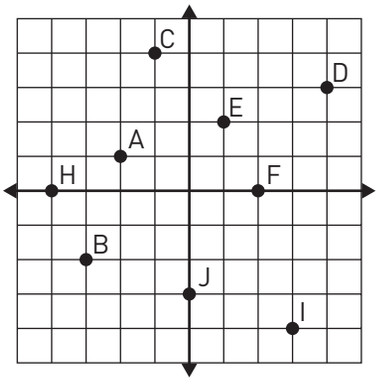
1. Start at the origin (0, 0).
2. Consider the x-coordinate.
   - If it is 0, remain at the origin.
   - If it is positive, move to the right along the x-axis. Stop at the x-coordinate.
   - If it is negative, move to the left along the x-axis. Stop at the x-coordinate.
3. Consider the y-coordinate.
   - If it is 0, graph the point where you stopped after finding the x-coordinate.
   - If it is positive, move up parallel to the y-axis, starting from the x-coordinate. Stop at the y-coordinate. Graph the point.
   - If it is negative, move down parallel to the y-axis, starting from the x-coordinate. Stop at the y-coordinate. Graph the point.

DIRECTIONS: State the letter that is near each ordered pair.

1. (1, 2) **2.** (−3, −2) **3.** (0, −3)

4. (−2, 1) **5.** (2, 0) **6.** (3, −4)

7. (−4, 0) **8.** (−1, 4) **9.** (4, 3)



CHALLENGE: Is (x, y) ever the same as (y, x)? Explain your answer.

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Teaching Notes 4.5: Completing T-Tables

The values in a t-table can be used to graph ordered pairs that can then be used to graph the equation of a line. Students often make mistakes substituting the value for the wrong variable in t-tables or they make mistakes solving the equation.

1. Explain that a t-table resembles a lowercase *t* and is sometimes used to organize ordered pairs. Note that the values for *x* are placed in the first column and the values for *y* are placed in the second column. This is similar to ordered pairs where *x* precedes *y*.
2. Explain that once students have substituted the correct value for the given variable, completing a t-table is similar to solving an equation.
3. Review the information and example on the worksheet. Note that students are asked to find the value of *y* when *x* is −1 and to find the value of *x* when *y* is −2. Point out the completed t-table that pairs −1 with −11 and 2 with −2.

EXTRA HELP:

You can check the values on your t-table by substituting the values for *x* and *y* into the equation. If the equation is true, you know that your values are correct.

ANSWER KEY:

(1) 4 **(2)** 8 **(3)** 5 **(4)** 5 **(5)** 3 **(6)** −1 **(7)** 2 **(8)** −2 **(9)** 6

(Challenge) The numbers are placed in the wrong columns. The values in the first column are the *y*-values, and the values in the second column are the *x*-values.

Name _____ Date _____

WORKSHEET 4.5: COMPLETING T-TABLES

Use the following steps to complete a t-table:

1. Determine the values that are provided in the t-table.
2. Substitute one of the given values of the variables in the equation.
3. Solve the equation to find the value of the other variable.
4. Place this value in the t-table.
5. Continue this process until the t-table is complete.

EXAMPLE

Complete the t-table for the equation $y = 3x - 8$.

x	y
-1	
	-2

Substitute -1 for x.

$y = 3x - 8$

$y = 3(-1) - 8$

$y = -3 - 8$

$y = -11$

Substitute -2 for y

$y = 3x - 8$

$-2 = 3x - 8$

$6 = 3x$

$2 = x$

Completed t-table

x	y
-1	-11
2	-2

DIRECTIONS: Complete each t-table.

$x + y = 5$

	x	y
1.	1	
2.		-3
3.		0

$3x = 4y + 1$

	x	y
4.	7	
5.		2
6.		-1

$y = 2x - 4$

	x	y
7.		0
8.	1	
9.		8

CHALLENGE: Find the error in this t-table, $2x + y = 10$.

x	y
8	1
10	0
12	-1

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Teaching Notes 4.6: Finding the Slope of a Line, Given Two Points on the Line

Using a formula for finding the slope of a line is relatively easy, provided students understand the formula. Most mistakes occur when students substitute incorrect values for the variables or make errors in computation.

SPECIAL MATERIALS:

Graph paper, rulers

1. Ask your students to draw a straight line through points (1, 7) and (−1, 4) on graph paper.
2. Explain that they will use their graph to find the slope, or "steepness," of the line through these points. Start with point (1, 7) and ask them how they can get to (−1, 4) by first moving vertically and then horizontally along the grid lines of the graph paper. (Move down 3 units and move 2 units to the left.) Remind your students that moving down is described as moving in a negative direction and moving to the left is also moving in a negative direction. The slope of the line through these points is $\frac{\text{vertical movement}}{\text{horizontal movement}}$ which is equal to $\frac{-3}{-2}$ or $\frac{3}{2}$.
3. Instruct your students to start with point (−1, 4) and ask them how they can get to (1, 7) using the same procedure. (Move up 3 units and move 2 units to the right.) Note that moving up is moving in a positive direction and moving right is also positive. Also note that the slope of the line is the same, $\frac{3}{2}$, regardless of the starting point.
4. Review the formula, information, and example on the worksheet with your students. Note that the example is completed in two different ways.

EXTRA HELP:

Using the formula is the most efficient way to find the slope of a line.

ANSWER KEY:

(1) −4 **(2)** −2 **(3)** $\frac{1}{4}$ **(4)** $\frac{4}{3}$ **(5)** 0

(Challenge) Disagree. The correct slope is $\frac{4}{3}$. Dominic placed the *x*-values in the numerator instead of the denominator.

Name _____ Date _____

WORKSHEET 4.6: FINDING THE SLOPE OF A LINE, GIVEN TWO POINTS ON THE LINE

The slope (m) of a line through two points can be found by using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. (x_1, y_1) and (x_2, y_2) are two points on the line. $x_2 \neq x_1$ Follow the steps below to find the slope of a line when given two points on the line:

1. Identify one point as (x_1, y_1) and the other as (x_2, y_2).
2. Substitute the values in the formula and solve for m.
3. Simplify the fractions and change improper fractions to whole numbers, if possible. (Do not change improper fractions to mixed numbers.)

EXAMPLE

Use the formula to find the slope of the line through $(-3, -1)$ and $(-2, 8)$.

$x_1 = -3, y_1 = -1, x_2 = -2, y_2 = 8$ or $x_1 = -2, y_1 = 8, x_2 = -3, y_2 = -1$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{8 - (-1)}{-2 - (-3)} = \frac{9}{1} = 9 \qquad m = \frac{-1 - 8}{-3 - (-2)} = \frac{-9}{-1} = 9$$

DIRECTIONS: Find the slope of the line through the given points.

1. $(-1, 9), (0, 5)$
2. $(0, 0), (2, -4)$
3. $(9, 5), (-7, 1)$
4. $(3, -4), (9, 4)$
5. $(2, 6), (4, 6)$

CHALLENGE: Dominic said that the slope of the line between $(-1, 0)$ and $(2, 4)$ is $\frac{3}{4}$. Do you agree? If you disagree, find his error and the correct slope.

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Teaching Notes 4.7: Identifying the Slope and *Y*-Intercept from an Equation

Two problems often arise when students are asked to identify the slope and *y*-intercept from an equation. First, they may misinterpret the formula, and second, they may make an error rewriting the equation.

1. Review the slope-intercept form of an equation, $y = mx + b$. Ask your students to identify the slope and *y*-intercept using equations such as $y = 3x - 1$ in which $m = 3, b = -1$; $y = -2x$ in which $m = -2, b = 0$; and $y = \frac{2}{5}x - \frac{1}{3}$ in which $m = \frac{2}{5}, b = \frac{-1}{3}$.
2. Explain that sometimes students must rewrite an equation to express it in slope-intercept form. They should do the following:

   - Isolate the *y*-variable, usually by adding a variable expression or a number to or subtracting a variable expression or a number from both sides.
   - Divide both sides of the equation by the coefficient of *y* (unless the coefficient is 1).

   Once the equation is rewritten in slope-intercept form, it is easy to find the slope and *y*-intercept by looking at the equation.
3. Review the information and examples on the worksheet with your students. Explain that because the second example is not written in slope-intercept form, it must be rewritten.

EXTRA HELP:

If the slope is an improper fraction, do not change it to a mixed number.

ANSWER KEY:

(1) $m = 3; b = -2$ **(2)** $m = 3; b = 0$ **(3)** $m = \frac{1}{2}; b = -6$ **(4)** $m = -1; b = 1$

(5) $m = \frac{3}{4}; b = 1$ **(6)** $m = 2; b = 3$ **(7)** $m = \frac{4}{5}; b = -2$ **(8)** $m = -\frac{7}{3}; b = \frac{10}{3}$

(Challenge) The *y*-intercept of each of the graphs is 0.

Name _____ Date _____

WORKSHEET 4.7: IDENTIFYING THE SLOPE AND *Y*-INTERCEPT FROM AN EQUATION

When you are given the equation of a line written in slope-intercept form, $y = mx + b$, m stands for the slope and b stands for the y-intercept. If the equation is not written in slope-intercept form, you must rewrite the equation to isolate the variable y.

Example 1: Find the slope and y-intercept of the graph of $y = 7x - 1$. The equation is written in slope-intercept form. The slope is 7 and the y-intercept is -1.

Example 2: Find the slope and y-intercept of the graph of $3x + 2y = 10$. This equation must be rewritten in slope-intercept form.

$$\begin{aligned} 3x + 2y &= 10 \\ -3x \qquad &\quad -3x \\ 2y &= -3x + 10 \\ \frac{2y}{2} &= \frac{-3x}{2} + \frac{10}{2} \\ y &= -\frac{3}{2}x + 5 \end{aligned}$$

The slope is $-\frac{3}{2}$; the y-intercept is 5.

DIRECTIONS: Find the slope and y-intercept of each equation.

1. $y = 3x - 2$

2. $y = 3x$

3. $y = \frac{1}{2}x - 6$

4. $y + x = 1$

5. $y = \frac{3}{4}x + 1$

6. $y - 2x - 3 = 0$

7. $4x - 5y = 10$

8. $3y + 7x = 10$

CHALLENGE: The graphs of the lines $y = x$, $2x + 3y = 0$, $y = 7x$, and $x = 7y$ all have something in common. What is it?

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Teaching Notes 4.8: Using Equations to Find the Slopes of Lines

The concept of slope is difficult for many students to master. Especially confusing are the slopes of horizontal and vertical lines. Students often confuse a slope of 0 with no slope as well as which type of line has no slope and which lines have a slope of 0.

1. Explain the formula that is used to find the slope of a line: $m = \frac{y_2 - y_1}{x_2 - x_1}$. Ask your students to find the slope of a line through points (3, 4) and (6, 8). In this case, the slope is 2.
2. Draw a horizontal line; for example, $y = 4$. Ask your students to select two points on the line. The coordinates of the points may vary but the y-values are the same. As students apply the slope formula, they will note that the numerator is equal to 0. This is true for all horizontal lines. Therefore the slope of a horizontal line is 0.
3. Draw a vertical line; for example, $x = 3$. Ask your students to select two points on the line. The coordinates of the points may vary but the x-values are the same. As students apply the slope formula, the denominator is equal to 0. This is true for all vertical lines. Emphasize that because division by 0 is undefined, a vertical line has no slope.
4. Review the information on the worksheet with your students. Note that some equations may have to be rewritten so that they are expressed in slope-intercept form.

EXTRA HELP:

Think of a horizontal line as flat, not steep. The slope of the line is 0.

ANSWER KEY:

(1) $m = 2$ **(2)** No slope **(3)** $m = 0$ **(4)** $m = 0$

(5) $m = -1$ **(6)** No slope **(7)** $m = -2$ **(8)** $m = 1$

(Challenge) Answers may vary. Any equation of the form $y = mx + 6$ is correct, provided that the slope is not equal to 0.

Name _____ Date _____

WORKSHEET 4.8: USING EQUATIONS TO FIND THE SLOPES OF LINES

Use the guidelines below to find the slopes of different kinds of lines:

Horizontal lines ↔	**Vertical lines** ↕	**Lines that are neither horizontal nor vertical**
• The slope is equal to 0. • The equation is $y = b$ where b is the y-intercept. • All y-coordinates are the same. • The graph is parallel to the x-axis.	• There is no slope. • The equation is $x = k$ where k is a constant. • All x-coordinates are the same. • The graph is parallel to the y-axis.	• The equation is $y = mx + b$ where m = slope and $b = y$-intercept.

DIRECTIONS: Find the slope of the line, if it exists.

1. $y = 2x + 7$

2. $x = 6$

3. $y = 4$

4. $y - 3 = 10$

5. $x + y = 4$

6. $x + 3 = 0$

7. $2x + y = 15$

8. $x = y$

CHALLENGE: The graph of the equation $y = 6$ has a y-intercept that is equal to 6. Write another equation of a line whose graph has a y-intercept that is equal to 6.

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Teaching Notes 4.9: Identifying Parallel and Perpendicular Lines, Given an Equation

Identifying parallel and perpendicular lines when given an equation is relatively easy. A common mistake arises from not understanding how the slopes of lines are related.

1. Draw and label the lines of $y = 3x + 1$ and $y = 3x - 2$ on the same graph as an example for your students. Explain that these lines are parallel because the distance between them does not change. Note that the slope of each line is 3. Emphasize that if two lines have the same slope, they are parallel.
2. Draw and label the lines of $y = -\frac{1}{3}x + 1$ and $y = 3x - 2$ on the same graph. Explain that these lines are perpendicular because they intersect at right angles. Ask your students how the slopes of the lines are related. (3 and $\frac{1}{3}$ are reciprocals; 3 and $-\frac{1}{3}$ have opposite signs.) Explain that if two lines have slopes that are reciprocals and have opposite signs, the lines are perpendicular. If necessary, provide more examples.
3. Provide an example of the graph of the intersection of a vertical line and a horizontal line, such as $y = 3$ and $x = 2$. Note that these two lines are perpendicular because each line is parallel to an axis ($y = 3$ is parallel to the x-axis and $x = 2$ is parallel to the y-axis) and the axes are perpendicular.
4. Review the information about parallel and perpendicular lines on the worksheet with your students.

EXTRA HELP:

Be sure that the equation is in slope-intercept form so that you may easily identify the slope. If the equation is not written in slope-intercept form, rewrite it.

ANSWER KEY:

(1) Parallel **(2)** Perpendicular **(3)** Neither **(4)** Perpendicular **(5)** Parallel **(6)** Perpendicular

(Challenge) Cheryl's conclusion is incorrect. The lines are not perpendicular. If they were, the slopes would have opposite signs (which they do), but the slopes would also be reciprocals.

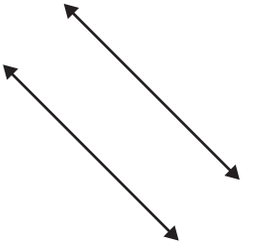
Name _____ Date _____

WORKSHEET 4.9: IDENTIFYING PARALLEL AND PERPENDICULAR LINES, GIVEN AN EQUATION

To find out if two lines are parallel or perpendicular, write their equations in slope-intercept form. Then find the slope. If the slopes are the same, the lines are parallel. If the slopes have different signs and are reciprocals, the lines are perpendicular. Following are facts about parallel and perpendicular lines:

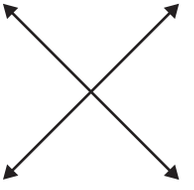
PARALLEL LINES

- Parallel lines have the same slope.
- They do not intersect.
- All horizontal lines are parallel.
- All vertical lines are parallel.



PERPENDICULAR LINES

- The slopes have opposite signs and are reciprocals.
- Perpendicular lines intersect at right angles.
- A vertical line and a horizontal line are always perpendicular.



DIRECTIONS: Find whether each pair of lines is parallel, perpendicular, or neither.

1. $y = 2x + 1; y = 2x - 4$

2. $y = x + 7; y = -x + 2$

3. $3y = 2x + 1; 4y = 2x - 3$

4. $y = -2x + 1; y = \frac{1}{2}x - 4$

5. $y = -2x; 2y + 4x = 5$

6. $y = -\frac{1}{4}x;\ y = 4x$

CHALLENGE: Cheryl looked at the equations $y = 4x + 1$ and $y = -2x + 1$. She said that the lines were perpendicular because the slopes have opposite signs. Do you agree with her conclusion? Why or why not?

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Teaching Notes 4.10: Using the *X*-Intercept and the *Y*-Intercept to Graph a Linear Equation

By finding the *x*-intercept and the *y*-intercept, students can quickly graph a linear equation. Typical errors result from not understanding the process for solving a linear equation and graphing ordered pairs.

SPECIAL MATERIALS:

Graph paper, rulers

1. Instruct your students to graph the equation $x + 2y = 6$ in the coordinate plane by providing the points $(-4, 5)$, $(4, 1)$, $(-2, 4)$, $(6, 0)$, $(0, 3)$, and $(8, -1)$. They should graph the points and see that they lie on a line.
2. Because two points determine a line, students need to find only two solutions to a linear equation in order to graph the equation. Two solutions that are easy to find are the *x*—intercept and the *y*-intercept.
3. Explain to your students that the points (6, 0) and (0, 3) in the example respectively represent the *x*-intercept and the *y*-intercept. The *x*-intercept is the point where the graph crosses the *x*-axis. The *y*-intercept is the point where the graph crosses the *y*-axis.
4. Review the information and example on the worksheet with your students. Model how to find these intercepts algebraically using the equation $x + 2y = 6$.
5. Explain to your students that after they find the intercepts of the equation, they can use these ordered pairs to create a graph. They should plot the two points and draw a line connecting them. This line represents the graph of the equation.

EXTRA HELP:

As a check, always select a third ordered pair that is a solution to the equation. If the ordered pair is on the line, then the graph is correct.

ANSWER KEY:

The *x*-intercept is provided first and is followed by the *y*-intercept. Use the intercepts to check students' graphs.

(1) (4, 0); (0, 2)	**(2)** (6, 0); (0, 2)	**(3)** (−5, 0); (0, 10)	**(4)** (3, 0); (0, −15)
(5) (9, 0); (0, 3)	**(6)** (−5, 0); (0, 5)	**(7)** (−6, 0); (0, −3)	**(8)** (16, 0); (0, 2)

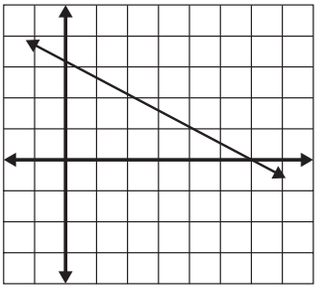
(Challenge) Marcus is correct. Horizontal or vertical lines will have only one intercept.

Name _____ Date _____

WORKSHEET 4.10: USING THE *X*-INTERCEPT AND THE *Y*-INTERCEPT TO GRAPH A LINEAR EQUATION

Graphs can be created in various ways. One way to create a graph is to use the *x*-intercept and the *y*-intercept.

EXAMPLE



To find the *x*-intercept of $x + 2y = 6$:

Replace *y* with 0 and solve for *x*.

$x + 2(0) = 6$

$x = 6$

The *x*-intercept is 6.

This represents the ordered pair (6, 0).

To find the *y*-intercept of $x + 2y = 6$:

Replace *x* with 0 and solve for *y*.

$0 + 2y = 6$

$2y = 6$

$y = 3$

The *y*-intercept is 3.

This represents the ordered pair (0, 3).

DIRECTIONS: Find the intercepts of each equation. Then use the intercepts to graph the equation.

1. $2x + 4y = 8$

2. $x + 3y = 6$

3. $y - 2x = 10$

4. $5x - y = 15$

5. $2x + 6y = 18$

6. $5y - 5x = 25$

7. $-10x - 20y = 60$

8. $\frac{1}{2}x + 4y = 8$

CHALLENGE: Marcus said that some lines will not have both an *x*-intercept and *y*-intercept. Do you agree with him? If you do, give an example of when this could be true.

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Teaching Notes 4.11: Using Slope-Intercept Form to Graph the Equation of a Line

A linear equation written in slope-intercept form, $y = mx + b$, can easily be used to create a graph. The slope of the line is m and the y-intercept is b. Students may confuse these two variables or inaccurately use the slope to graph the equation of the line.

1. Explain that slope-intercept form, $y = mx + b$, is one way to write an equation of a line. m represents the slope of the line and b represents the y-intercept.
2. Present the equation $y = 3x + 2$. Have your students identify the slope and y-intercept. (The slope is 3 and the y-intercept is 2.)
3. Show your students how to graph this equation. Begin by plotting point (0, 2), which is the y-intercept. Then use the slope to plot the next point. Because the slope is 3, which can be expressed as $\frac{3}{1}$, students would move up three units and one unit to the right. Also, because 3 may be expressed as $\frac{-3}{-1}$, students could move down three units and one unit to the left. These movements are commonly referred to as $\frac{\text{vertical movement}}{\text{horizontal movement}}$ or $\frac{\text{rise}}{\text{run}}$.
4. Review the information on the worksheet with your students.

EXTRA HELP:

Always express the slope as a fraction. It is easier to apply rise over run.

ANSWER KEY:

(1) A, slope = 3, y-intercept = 4 **(2)** F, slope = −1, y-intercept = −1
(3) D, slope = 2, y-intercept = 1 **(4)** B, slope = 1.5, y-intercept = 3
(5) C, slope = 2, y-intercept = −1 **(6)** E, slope = −3, y-intercept = 4

(Challenge) Both equations represent the same slope and y-intercept. Joe's equation is written in slope-intercept form, and Brian's is written in standard form. However, each equation represents the same line.

Name _____ Date _____

WORKSHEET 4.11: USING SLOPE-INTERCEPT FORM TO GRAPH THE EQUATION OF A LINE

A linear equation is expressed in slope-intercept form when it is written as $y = mx + b$. m represents the slope of the line and b represents the y-intercept. Follow the steps below to graph a line using slope-intercept form:

1. Identify the slope and y-intercept.
2. Graph the y-intercept.
3. Mark the next point on the graph using the slope expressed as $\frac{\text{rise}}{\text{run}}$. If the numerator is positive, move up. If it is negative, move down. If the denominator is positive, move right. If it is negative, move left.

DIRECTIONS: Match each equation with the appropriate line on the diagram below. Write the slope and y-intercept of each equation.

1. $y = 3x + 4$

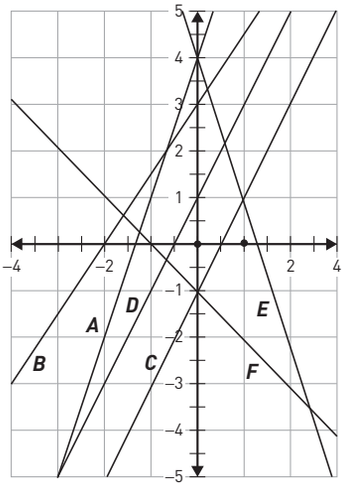
2. $y = -x - 1$

3. $y = 2x + 1$

4. $y = 1.5x + 3$

5. $y = 2x - 1$

6. $y = -3x + 4$



CHALLENGE: Joe has been using slope-intercept form to graph lines. He graphed a line using the equation $y = 4x + 2$. His friend Brian used the equation $-4x + y = 2$ to graph a line. When they compared their graphs, the lines were the same. How could this be? Explain.

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Teaching Notes 4.12: Graphing Linear Inequalities in the Coordinate Plane

Once students have mastered graphing equations in the coordinate plane, they are ready to graph linear inequalities. Deciding whether to make a solid line or a dotted line and shading the correct part of the graph often prove to be problem areas.

1. Explain the basic steps for graphing a linear equation. Especially note the following:
   - Students may use the *x*- and *y*-intercepts to graph a linear equation or they may graph the *y*-intercept and then move vertically and horizontally, depending on the slope, to graph the second point.
   - Students should draw a line connecting the points, understanding that all solutions to the equation are on the line.
2. Note that the graph of a linear equation separates the coordinate plane into three parts: a set of points above the line, the points on the line, and the points below the line. The graph of a linear inequality is shown by shading the portion of the coordinate plane that is above or below the line. The boundary line may or may not be a part of the graph.
3. Review the information and example on the worksheet with your students, particularly the meaning of the inequality symbols as they apply to graphing.

EXTRA HELP:

Select a point in the portion of the graph that you have shaded to be sure it is a solution of the inequality. If it is not a solution, shade the other portion of the graph.

ANSWER KEY:

The *x*-intercept, *y*-intercept, and a description of the graph are provided.

(1) (−2, 0); (0, 2); dotted line, shaded below
(2) (0, 0); (0, 0); solid line, shaded above
(3) (0, 0); (0, 0); solid line, shaded below
(4) (−4, 0); (0, 2); dotted line, shaded below
(5) (1, 0); (0, −1); dotted line, shaded below
(6) (−2, 0); (0, 4); solid line, shaded above
(7) (0, 0); (0, 0); dotted line, shaded below
(8) (4, 0); (0, −4); solid line, shaded above

(Challenge) Disagree. The graph of $x < 4$ is a dotted vertical line that crosses the *x*-axis at (4, 0). The solutions to this inequality include all the values to the left of the line.

Name _____ Date _____

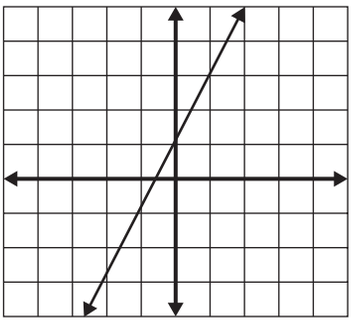
WORKSHEET 4.12: GRAPHING LINEAR INEQUALITIES IN THE COORDINATE PLANE

Use the guidelines below for graphing inequalities:

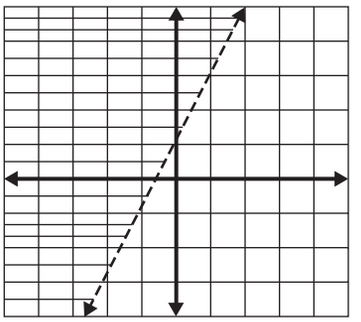
- The symbol $>$ means that all the points above the boundary line are solutions to the inequality. Draw a dotted boundary line and shade above it.
- The symbol $\geq$ means that all the points on the boundary line and the points above it are solutions to the inequality. Draw a solid boundary line and shade above it.
- The symbol $<$ means that all the points below the boundary line are solutions to the inequality. Draw a dotted boundary line and shade below it.
- The symbol $\leq$ means that all the points on the boundary line and the points below it are solutions to the inequality. Draw a solid boundary line and shade below it.

EXAMPLE

Comparisons of the graphs $y = 2x + 1$ and $y > 2x + 1$.



$y = 2x + 1$



$y > 2x + 1$

DIRECTIONS: Graph each inequality.

1. $y < x + 2$

2. $y \geq -2x$

3. $y \leq x$

4. $2y - x < 4$

5. $-y + x > 1$

6. $-2x + y \geq 4$

7. $x > y$

8. $x - 4 \leq y$

CHALLENGE: Sam says that $x < 4$ cannot be graphed because it is impossible to shade below the graph of $x = 4$. Do you agree? Explain your reasoning.

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Teaching Notes 4.13: Writing a Linear Equation, Given Two Points

Writing a linear equation when two points are given is often a difficult task for students. It requires background knowledge for finding the slope, solving equations, and using slope-intercept form.

1. Explain to your students that they can use two points to write an equation of a line. The first step is to find the slope of a line. Depending on the abilities of your students, you may find it helpful to review the slope formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$
2. Review the slope-intercept form, $y = mx + b$, where m stands for the slope and b stands for the y-intercept. Explain that students will substitute values for m, x, and y, and then solve for b.
3. Note that the final equation should be written in slope-intercept form with variables for x and y, and numbers for m and b.
4. Review the information and example on the worksheet with your students. Emphasize that either ordered pair could be substituted for x and y. If necessary, discuss the steps for solving the equations.

EXTRA HELP:

Check your work by creating a graph. Plot the given two points and draw a line through them. This line should have the same slope and y-intercept as your equation.

ANSWER KEY:

(1) $y = 4x + 5$ **(2)** $y = -2x + 3$ **(3)** $y = 5x - 10$ **(4)** $y = -x - 2$

(5) $y = \frac{1}{2}x + 4$ **(6)** $y = \frac{2}{3}x - 3$ **(7)** $y = -\frac{1}{4}x + 6$ **(8)** $y = -7x + 17$

(Challenge) They are both correct. The equations $y = x$ and $y = x + 0$ are equivalent, although $y = x$ is the preferred form.

Name _____ Date _____

WORKSHEET 4.13: WRITING A LINEAR EQUATION, GIVEN TWO POINTS

Follow the steps below to write an equation from two points on a graph:

1. Find the slope of the line. Use the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$.
2. Find the y-intercept. Use the equation $y = mx + b$. Replace m with the slope of the line. Replace x and y with the values from one of the given ordered pairs. Solve for b.
3. Write the equation of the line using slope-intercept form. Substitute the slope, m, and the y-intercept, b, into slope-intercept form, $y = mx + b$.

EXAMPLE

Write an equation of the line that passes through the points (3, 1) and (6, 7).

Find the slope of the line: $m = \frac{7 - 1}{6 - 3} = \frac{6}{3} = 2$. The slope of the line is 2.

Find the y-intercept: $1 = 2(3) + b$ or $1 = 6 + b$. The y-intercept is -5.

The equation of the line is $y = 2x - 5$.

DIRECTIONS: Write an equation in slope-intercept form of the line that passes through the points.

1. (1, 9), (2, 13)
2. (5, −7), (3, −3)
3. (3, 5), (−2, −20)
4. (1, −3), (−1, −1)
5. (2, 5), (8, 8)
6. (3, −1), (9, 3)
7. (12, 3), (−4, 7)
8. (2, 3), (4, −11)

CHALLENGE: Bob found the equation of the line that contains points (1, 1) and (5, 5) to be $y = x$. Jim said that the equation for this line should be written as $y = x + 0$ because 0 is the y-intercept. Who is correct? Explain.

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Teaching Notes 4.14: Finding the Equation of the Line of Best Fit

The line of best fit is a line that can be drawn through points on a scatter plot to show a trend between two sets of data. Many students find this concept abstract. Drawing a line suggested by the points enables students to determine the equation of the line; however, the fact that there is no exact answer can cause confusion.

SPECIAL MATERIALS:

Graph paper, rulers

1. Explain that in real life data may not always lie in a straight line. In such cases, students have to draw a line of best fit on a scatter plot that approximates the data. Not all of the points will be on the line.
2. Offer an example of movie ticket prices students may purchase at various theaters. The number of tickets represents the x-coordinate and the total paid for the tickets represents the y-coordinate. An example of an ordered pair might be (2, \$12.00), which means that two tickets cost \$12.00. Ask your students to create a scatter plot by graphing the following ordered pairs: (2, \$12.00), (1, \$7.00), (1, \$6.50), (2, \$9.00), (4, \$28.00), (1, \$7.50), (3, \$16.50), (4, \$30.00), and (5, \$25.00). Instruct them to draw the line of best fit and find the equation of the line. Depending on their abilities, you might find it helpful to review 4.13: "Writing a Linear Equation, Given Two Points." Because the lines of best fit of students may vary, the equations may also vary. One possible answer is $y = 6x + 0.6$.
3. Review the information on the worksheet with your students. Emphasize that the first number of an ordered pair represents the x-coordinate and the second represents the y-coordinate. Remind students that they must select two points on their line to find the slope. If necessary, review rounding to the nearest tenth.

EXTRA HELP:

A line of best fit is also called the "best-fitting line."

ANSWER KEY:

Answers will vary. Possible answers follow.

(1) $y = 1.6x + 0.8$ **(2)** $y = 0.7x - 0.1$ **(3)** $y = 3.3x - 1.1$ **(4)** $y = 0.5x + 0.3$

(Challenge) There is no way to tell if her equation would be very different. It depends on how close those points were to her line.

Name _____ Date _____

WORKSHEET 4.14: FINDING THE EQUATION OF THE LINE OF BEST FIT

A line of best fit is a line that approximates a relationship between a group of ordered pairs. Use the steps below to find the equation of the line of best fit:

1. Graph the ordered pairs to create a scatter plot.
2. Use a ruler to draw a line of best fit. Some points may be on your line and others may not.
3. Choose two points on the line you have drawn. Use the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ to find the slope of the line.
4. Find the y-intercept by substituting the x-coordinate of one point, the y-coordinate of the same point, and the slope in the formula $y = mx + b$. Then solve for b.
5. Write the equation of the line of best fit in slope-intercept form.

DIRECTIONS: Find the equation of the line of best fit for the data below. Round to the nearest tenth.

1. (3, 4), (2, 7), (5, 8), (6, 10), (4, 8), (5, 9.5), (8, 15), (1, 1.5), (8, 13), (3, 5)

2. (10, 6), (8, 4), (20, 10), (15, 12), (14, 12), (9, 4), (15, 10), (20, 15), (4, 3), (6, 5)

3. (1, 3), (1, 4), (1, 2), (2, 4), (2, 5), (3, 6), (3, 7), (3, 10), (4, 15), (4, 12)

4. (10, 5), (1, 0.75), (8, 5), (6, 4), (7, 3), (4, 1.5), (9, 4), (8, 6), (1, 1), (2, 1.5)

CHALLENGE: Michelle drew a line of best fit. But instead of choosing points on the line, she chose the first and last points of the data to write the equation. Will her equation be very different from the equation she would get if she had chosen two points on the line? Explain your answer.

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Teaching Notes 4.15: Using the Midpoint Formula

A midpoint is a point on a line segment that is equidistant from its endpoints. The midpoint between (x_1, y_1) and (x_2, y_2) can be found by using the midpoint formula: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. Although this formula seems relatively straightforward, students may replace the variables with the wrong value, resulting in an incorrect midpoint.

SPECIAL MATERIALS:

Graph paper, rulers

1. Explain that a midpoint is the point on a line segment that divides the segment into two equal parts. It is written as an ordered pair.
2. Present the midpoint formula and relate it to averages. To find the average of two numbers, you must add the numbers and divide by two. The average of these numbers is the number exactly between the two numbers being averaged. Similarly, the midpoint formula takes the average of the *x*-coordinates and the average of the *y*-coordinates of the two ordered pairs.
3. Review the steps for finding the midpoint and the example on the worksheet with your students. Explain that students must label the ordered pairs as (x_1, y_1) or (x_2, y_2) in the problem and correctly replace each variable with the appropriate value. Ask your students to verify that (4, 5) is the midpoint by graphing this point and the segment whose endpoints are (2, 3) and (6, 7) on graph paper.

EXTRA HELP:

Rewrite the formula with the appropriate *x*- and *y*-coordinates before you solve. Taking this extra step will help eliminate careless mistakes.

ANSWER KEY:

(1) $\left(3\frac{1}{2}, 4\frac{1}{2}\right)$ **(2)** (3, 5) **(3)** (1, 3) **(4)** $\left(1, -1\frac{1}{2}\right)$

(5) (−6, −2) **(6)** $\left(\frac{1}{2}, -3\right)$ **(7)** (2, 7) **(8)** $\left(\frac{1}{2}, 3\frac{1}{2}\right)$

(Challenge) The missing *x*-coordinate is 9.

Name _____ Date _____

WORKSHEET 4.15: USING THE MIDPOINT FORMULA

A midpoint is a point on a line segment located exactly between the two endpoints. The midpoint between two points, (x_1, y_1) and (x_2, y_2), can be found by using the midpoint formula: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. Follow the steps below to find the midpoint between two points:

1. Determine which ordered pair will represent (x_1, y_1) and which will represent (x_2, y_2).
2. Replace each variable in the midpoint formula with the appropriate x-coordinates or y-coordinates.
3. Solve to find each value.

EXAMPLE

Find the midpoint between (2, 3) and (6, 7).

Label (2, 3) as (x_1, y_1) and (6, 7) as (x_2, y_2).

Replace each variable in the midpoint formula with the correct values: $\left(\frac{2 + 6}{2}, \frac{3 + 7}{2}\right)$.

The midpoint between (2, 3) and (6, 7) is (4, 5).

DIRECTIONS: Find the midpoint between each pair of points.

1. (3, 0), (4, 9)
2. (1, 2), (5, 8)
3. (−1, 2), (3, 4)
4. (−3, −6), (5, 3)
5. (−4, −1), (−8, −3)
6. (10, −9), (−9, 3)
7. (1, 7), (3, 7)
8. (0, 2), (1, 5)

CHALLENGE: Will's dog took a bite out of his homework and he lost part of his work. All that remained was the ordered pair (3, 4), the midpoint (6, 2), and the y-coordinate of 0 for the second ordered pair. Use the midpoint formula to find the missing x-coordinate.

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Teaching Notes 4.16: Using the Distance Formula to Find the Distance Between Two Points

Many students find the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, intimidating because of the subscripts, squares, and square root. A fundamental understanding of this notation enables most students to use this formula correctly.

1. Refer to the diagram on the worksheet, noting that the distance between (1, 4) and (2, 3) is the length of the hypotenuse of the right triangle. Note that the length of each leg is one unit. Explain that the Pythagorean theorem states that in a right triangle $a^2 + b^2 = c^2$ where c is the hypotenuse. Because $a = 1$ and $b = 1$, $1^2 + 1^2 = c^2$, therefore $c = \sqrt{2}$, which is about 1.414.
2. Explain that it is inefficient to make a diagram and use the Pythagorean theorem to find the distance between two points. The distance formula may be used instead.
3. Present the distance formula along with a coordinate plane on which the points (1, 4) and (2, 3) are plotted. (These points also serve as the example on the worksheet.)
4. Explain the meaning of each variable in the distance formula. d represents the distance between the points. x_2 represents the value of the x-coordinate in the second ordered pair. x_1 represents the value of the x-coordinate in the first ordered pair. Similarly, y_2 and y_1 represent the values of the y-coordinate in the second and first ordered pair, respectively.
5. Review the information and example on the worksheet with your students. Explain the steps for using the distance formula to find the distance between (1, 4) and (2, 3) as shown in the example. Depending on their abilities, you may find it necessary to review the order of operations and rounding to the nearest hundredth. See 1.5: "Simplifying Expressions with Grouping Symbols and Exponents."

EXTRA HELP:

Simplify the expression inside the radical symbol first. Then find the square root.

ANSWER KEY:

(1) $\sqrt{29} \approx 5.39$ **(2)** $\sqrt{20} \approx 4.47$ **(3)** $\sqrt{34} \approx 5.83$ **(4)** $\sqrt{13} \approx 3.61$

(5) $\sqrt{8} \approx 2.83$ **(6)** 10 **(7)** $\sqrt{17} \approx 4.12$ **(8)** $\sqrt{5} \approx 2.24$

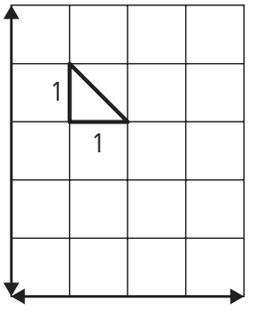
(Challenge) Disagree. Sammie's method is the shortest way to find the distance between two points on a vertical line. However, she could have also used the distance formula to obtain the same answer.

Name _____ Date _____

WORKSHEET 4.16: USING THE DISTANCE FORMULA TO FIND THE DISTANCE BETWEEN TWO POINTS

To find the distance between two points, follow the steps below:

1. Use the distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. Replace each variable with the correct x- or y-coordinate.
3. Solve using the order of operations.



EXAMPLE

Find the distance between (1, 4) and (2, 3). In this problem, (1, 4) represents the first ordered pair (x_1, y_1) and (2, 3) represents the second ordered pair (x_2, y_2).

$$d = \sqrt{(2-1)^2 + (3-4)^2}$$
$$d = \sqrt{(1)^2 + (-1)^2}$$
$$d = \sqrt{1+1}$$
$$d = \sqrt{2} \approx 1.414$$

Rounded to the nearest hundredth, the distance between (1, 4) and (2, 3) is about 1.41.

DIRECTIONS: Use the distance formula to find the distance between each pair of points. Round to the nearest hundredth if necessary.

1. (2, 0), (7, 2)
2. (6, 1), (8, 5)
3. (3, −1), (0, 4)
4. (5, 4), (8, 2)
5. (−4, −5), (−2, −3)
6. (−2, −3), (6, 3)
7. (2, −7), (3, −3)
8. (−2, 3), (−1, 5)

CHALLENGE: Sammie said that the distance formula cannot be used to find the distance between two points on a vertical line. Instead, she said that you can subtract the y-coordinates of two points on the vertical line. For example, (0,3) and (0,10) are 7 units apart because $10 - 3 = 7$. Do you agree with her? Explain your answer.

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Teaching Notes 4.17: Graphing Systems of Linear Equations When Lines Intersect

Systems of linear equations can be solved in several ways, one of which is by graphing. To successfully solve systems of linear equations by graphing, students must first understand how to graph equations and recognize the point of intersection as the solution to the system.

1. Explain that one way to solve a system of linear equations is to graph each equation and find the point of intersection.
2. Explain the basic steps for graphing a linear equation:
   - Students may use the *x*- and *y*-intercepts to graph a linear equation or they may graph the *y*-intercept and then move vertically and horizontally, depending on the slope, and then graph the second point.
   - Students should draw a line connecting the two points.
3. Review the information and example on the worksheet with your students. Note that a solution to a system of equations is represented by the point where the graphs intersect, an ordered pair (x, y) that satisfies each equation in the system.

EXTRA HELP:

Check solutions algebraically by substituting each value into the original system of equations.

ANSWER KEY:

(1) (2, 12) **(2)** (−5, 1) **(3)** (1, 1) **(4)** (−2, 7) **(5)** (2, −1) **(6)** (−4, 5) **(7)** (10, 5) **(8)** (3, −5)

(Challenge) The graphing method may still be used to solve systems of equations when the solutions are not integers; however, it is not the best method because the values for *x* and *y* must be estimated. Substitution, addition or subtraction, or multiplication with addition or subtraction, would be better methods to solve these problems.

Name _____ Date _____

WORKSHEET 4.17: GRAPHING SYSTEMS OF LINEAR EQUATIONS WHEN LINES INTERSECT

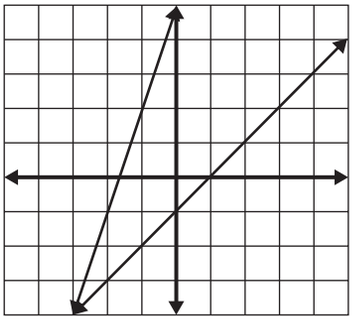
Follow the steps below to find the solutions to systems of linear equations by graphing:

1. Write each equation in slope-intercept form (if it is not already in this form).
2. Graph both equations on the same coordinate plane.
3. Find the coordinates of the point of intersection. The point of intersection represents the solution to the system.
4. Check the coordinates algebraically by substituting each value into the original system of equations.

EXAMPLE

$y = 3x + 4$

$y = x - 2$



$(-3, -5)$ is the point of intersection, which represents the solution to the system.

DIRECTIONS: Solve each system of linear equations.

1. $y = 3x + 6$
$y = 2x + 8$

2. $y = x + 6$
$y = -2x - 9$

3. $y = 2x - 1$
$y = 5x - 4$

4. $y = -x + 5$
$y = 3x + 13$

5. $2y = -3x + 4$
$3y = x - 5$

6. $y = -2x - 3$
$y - x = 9$

7. $5y = 2x + 5$
$2y = x$

8. $y + 3x = 4$
$y + x = -2$

CHALLENGE: Mike solved a system of linear equations by graphing. The solutions were not integers. How might he have solved this system of equations? Explain your reasoning.

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Teaching Notes 4.18: Graphing Systems of Linear Equations if Lines Intersect, Are Parallel, or Coincide

The graphs of systems of linear equations may intersect, be parallel, or coincide. Each of these variations has different meanings for the solution of the system, causing confusion for many students.

1. Make sure that your students understand these terms: intersect, parallel, and coincide. Emphasize that the graphs of systems of linear equations may intersect, be parallel, or coincide.
2. Explain that when the graphs of the equations in a system intersect, the intersection is the solution to each equation in the system. See 4.17: "Graphing Systems of Linear Equations When Lines Intersect."
3. Explain that if the graphs of the equations do not intersect at one point, they may be parallel or they may coincide. If the lines are parallel, there is no solution. If the lines coincide, the graph is one line. In this case, every solution to the system of equations is on that line and the number of solutions is infinite.
4. Review the information on the worksheet with your students. Present the examples and graphs of these systems of equations: $y = 2x + 1, y = 3x$, which has one solution; $y = 3x + 4, y = 3x - 1$, which has no solution because the lines are parallel; and $y = 3x + 2, 2y = 6x + 4$, which has an infinite number of solutions. Note the slope and y-intercept of each graph.

EXTRA HELP:

Be sure you have rewritten the equation correctly.

ANSWER KEY:

(1) No solution

(2) One solution; $(-3, -9)$

(3) Infinite number of solutions; all are on the graph of $y = 2x - 1$

(4) One solution; (0, 0)

(5) No solution

(6) Infinite number of solutions; all are on the graph of $y = -x + 10$

(7) One solution; (6, 2)

(8) No solution

(Challenge) Disagree. If two lines have the same slope and different y-intercepts, there is no solution. If two lines have the same slope and the same y-intercept, there is an infinite number of solutions.

Name _____ Date _____

WORKSHEET 4.18: GRAPHING SYSTEMS OF LINEAR EQUATIONS IF LINES INTERSECT, ARE PARALLEL, OR COINCIDE

Follow the steps below to graph systems of linear equations:

1. Write the equation in slope-intercept form, if necessary.
2. Find the slope and y-intercept of each line.
3. Use the following facts to find the solutions, if any:
   - If the slopes are different and the y-intercepts are different, the lines intersect. There is one solution. It can by found by graphing or solving the system algebraically.
   - If the slopes are the same and the y-intercepts are different, the lines are parallel. There is no solution.
   - If the slopes are the same and the y-intercepts are the same, the lines coincide. There is an infinite number of solutions. Every solution to the system of equations is on the graph of the line.

DIRECTIONS: Solve each system by graphing. Find each solution, if possible, or write "no solution" or an "infinite number of solutions."

1. $y = -x + 4$
   $y = -x + 2$

2. $y = 4x + 3$
   $y = x - 6$

3. $3y = 6x - 3$
   $y = 2x - 1$

4. $y = x$
   $y = -x$

5. $2y = 4x$
   $y = 2x + 1$

6. $x + y = 10$
   $y = -x + 10$

7. $x + y = 8$
   $x - y = 4$

8. $4x + 2y = 10$
   $y = -2x$

CHALLENGE: Megan said that a system of equations that contains the equations of two lines that have the same slope always has no solution. Do you agree? Explain.

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SECTION 5

Monomials and Polynomials

Teaching Notes 5.1: Applying Monomial Vocabulary Accurately

Although the study of monomials is relatively easy, students often have trouble with the terminology. Consistent reinforcement of the terms can help clarify confusion.

1. Discuss the following facts with your students:
   - A monomial is a numeral (called a "constant"), a variable, or a product of a numeral and one or more variables. The exponent of each variable must be a positive integer.
   - The coefficient of the monomial is the numerical factor of the monomial (usually the first number). If a coefficient is not written, it is understood to be 1. For example, "1" is the coefficient of the monomial x.
   - The degree of each variable in a monomial is the number of times the variable occurs as a factor.
   - The degree of the monomial is the sum of the degrees of the variables in the monomial.
2. To summarize these facts, discuss the examples on the first chart on the worksheet with your students. Especially point out that $3x$ and $\frac{1}{5}x$ are examples of similar monomials because they have the same variable and degree.

EXTRA HELP:

Monomials are never sums or differences.

ANSWER KEY:

(1) 10, 5, 6, 11 **(2)** 1, 1, 1, 2 **(3)** −5, 3, 1, 4 **(4)** $-\frac{1}{2}$, 3, 2, 5 **(5)** $5xy$, 2 **(6)** $\frac{1}{2}x^3y^4$, 3

(Challenge) The commutative property of multiplication.

Name _____ Date _____

WORKSHEET 5.1: APPLYING MONOMIAL VOCABULARY ACCURATELY

Study the chart below, which provides examples of terms associated with monomials.

MONOMIAL	COEFFICIENT	DEGREE OF *X*	DEGREE OF *Y*	DEGREE OF MONOMIAL
$3x$	3	1	0	1 (the sum of 1 and 0)
x^2y^4	1	2	4	6 (the sum of 2 and 4)
$-x^5y^{10}$	-1	5	10	15 (the sum of 5 and 10)
$\frac{1}{5}x$	$\frac{1}{5}$	1	0	1 (the sum of 1 and 0)
6	6	0	0	0 (the sum of 0 and 0)

DIRECTIONS: Complete the following chart.

	MONOMIAL	COEFFICIENT	DEGREE OF *X*	DEGREE OF *Y*	DEGREE OF MONOMIAL
1.	$10\,x^5y^6$				
2.	xy				
3.	$-5x^3y$				
4.	$-\frac{1}{2}x^3y^2$				
5.		5	1	1	
6.		$\frac{1}{2}$		4	7

CHALLENGE: What property states that the order of the variables of similar monomials may be written in any order?

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Teaching Notes 5.2: Identifying Similar Terms

Understanding which terms are similar and which are not is an essential concept for adding and subtracting polynomials. When students have difficulty adding and subtracting polynomials, mistakes often arise from an inability to identify similar terms.

1. Explain that similar terms have the same variable to the same degree. Provide examples such as $3x$ and $-4x$, $9y$ and $3y$, $-5x^2$ and $8x^2$, xy and yx, and $5xy^3$ and xy^3. Note that similar terms differ only in their numerical coefficient. The order in which the variables are written does not matter. If necessary, review monomial vocabulary, which is the focus of 5.1: "Applying Monomial Vocabulary Accurately." Students need to understand terms such as "degree" and "coefficient" in order to complete Worksheet 5.2.
2. Provide examples of terms that are not similar, such as x and x^2, $2x$ and $2y$, and $3x^2$ and $3x^3$, to further reinforce the concept of similar terms. Ask your students to explain why these pairs of terms are not similar.
3. Review the examples on the worksheet with your students.

EXTRA HELP:

Every term has a coefficient. If none is written, the coefficient is 1.

ANSWER KEY:

Pairs of similar terms follow: $3x$ and $7x$; $\frac{1}{5}x^2$ and $-x^2$; $6a^2b$ and $\frac{1}{4}a^2b$; $6a$ and $-$a; $8ab$ and $2ba$; $3x^2y$ and $6x^2y$

(Challenge) They are similar. $3x$ is another way of expressing $3x^1$.

Name _____ Date _____

WORKSHEET 5.2: IDENTIFYING SIMILAR TERMS

Similar terms are exactly the same except for their coefficients.

EXAMPLES OF SIMILAR TERMS

$3x$ and $-x$; $7xy$ and $6yx$; $2\frac{3}{4}a^2$ and $6a^2$; $\frac{1}{2}x^2$ and $4x^2$; ab and $3ab$; $10ab$ and $-2ab$

EXAMPLES OF TERMS THAT ARE NOT SIMILAR

$3x$ and $3x^2$; $\frac{2}{3}a^2$ and $3ab$; $4a$ and $9b^2$; $4ab$ and $4a$; $-3x^2y$ and $-3xy^2$; $10a$ and $10ab$

DIRECTIONS: Every term is similar to another term below. Identify each pair of similar terms.

$3x$; $\frac{1}{5}x^2$; $-x^2$; $6a$; $8ab$; $7x$; $6a^2b$; $2ba$; $\frac{1}{4}a^2b$; $3x^2y$; $-a$; $6x^2y$

CHALLENGE: Are $3x$ and $3x^1$ similar terms? Explain your reasoning.

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Teaching Notes 5.3: Adding Polynomials

Adding polynomials is an important algebraic skill. A common error students make when adding polynomials is to add unlike terms.

1. Provide this example: A teacher asked two students to help collect worksheets and calculators at the end of the class. Two students volunteer to help. One collects 15 worksheets and 10 calculators and the other student collects 8 worksheets and 5 calculators. If w represents the number of worksheets and c represents the number of calculators, $(15w + 10c) + (8w + 5c)$ shows the number of items collected. Note that the parentheses enclose the items collected by each student. Explain that to find the sum (or total number of worksheets and calculators), your students should add the number of worksheets together and then add the number of calculators: $15w + 8w + 10c + 5c = 23w + 15c$ or 23 worksheets and 15 calculators. Emphasize that the sum is found by adding similar terms: the coefficients of the variable w were added and the coefficients of the variable c were added.
2. Explain that to add polynomials the coefficients of all of the similar terms must be added just as students would add integers. Also note that the numbers that have no variables are called "constants" and can be added with other constants, following the rules for adding integers. Depending on the abilities of your students, you may find it necessary to review the steps for adding integers.
3. Review the information and examples on the worksheet with your students. Make sure your students understand what constitutes similar terms.

EXTRA HELP:

Only the coefficients of similar terms can be added. If two terms are not similar, they cannot be added.

ANSWER KEY:

(1) $7x + 5$ **(2)** $2x^2 + x$ **(3)** $6a + 4b + 1$ **(4)** $4x^2 + x + 10$ **(5)** -7 **(6)** $3x^2 + xy + 2$ **(7)** $9a^2 - 5$ **(8)** $2y - 5xy - 11x$ **(9)** 4 **(10)** $4x^2 + 7x - 21$

(Challenge) The sum of polynomials will never contain any similar terms because similar terms must be combined when added.

Name _____ Date _____

WORKSHEET 5.3: ADDING POLYNOMIALS

Follow the steps below to find the sum of polynomials:

1. Add the coefficients of similar terms.
2. Add any constants (numbers that have no variables).

EXAMPLES

$(3x + 4) + (-2x + 7)$: Add $3x$ and $-2x$ because they are similar terms. Add 4 and 7 because they are constants. The sum is $x + 11$.

$(7x^2 + 2x - 3) + (-6x^2 + 8x - 9)$: Add $7x^2$ and $-6x^2$ because they are similar terms. Add $2x + 8x$ because they are similar terms. Add -3 and -9. The sum is $x^2 + 10x - 12$.

DIRECTIONS: Find the sum.

1. $(3x + 3) + (4x + 2)$
2. $(x^2 + 2x + 1) + (x^2 - x - 1)$
3. $(-2a + 3b) + (8a + b + 1)$
4. $(8x^2 + 3) + (-4x^2 + x + 7)$
5. $(x - 3) + (-x - 4)$
6. $(4x^2 + 3xy + 5) + (-x^2 - 2xy - 3)$
7. $(8a^2 - 9) + (a^2 + 4)$
8. $(3y + 2xy - 10x) + (-y - 7xy - x)$
9. $(3a + 6b - 4) + (-3a - 6b + 8)$
10. $(x^2 + 7x - 12) + (3x^2 - 9)$

CHALLENGE: Explain why the sum of polynomials will never contain any similar terms.

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Teaching Notes 5.4: Subtracting Polynomials

To subtract polynomials, students must master the concept of rewriting subtraction as addition and changing the sign of each of the terms in the polynomial they are subtracting. A common problem is forgetting to change the signs of the terms.

1. Provide this example: A teacher is distributing worksheets and calculators at the beginning of class. She has 25 worksheets and 25 calculators. She hands out 23 worksheets and provides 8 calculators for students who need them. If w represents the number of worksheets and c represents the number of calculators, $(25w + 25c) - (23w + 8c)$ represents the number of items that are left over. Explain that to find the difference students must rewrite the problem as $(25w + 25c) + (-23w + (-8c))$ and then add the polynomials. $25w + (-23w) + 25c + (-8c) = 2w + 17c$. Thus, the teacher has 2 worksheets and 17 calculators that she did not distribute.
2. Review the information and examples on the worksheet with your students. Emphasize that they must rewrite the problem by adding the opposite of each term of the polynomial they are subtracting. Make sure that your students recognize the steps involved in rewriting. After rewriting the polynomial to be subtracted, they must combine similar terms by adding the coefficients as they would add integers. Also note that constants must be added, following the rules for adding integers. If necessary, review the steps for adding and subtracting integers.

EXTRA HELP:

The opposite of a positive coefficient is a negative coefficient and the opposite of a negative coefficient is a positive coefficient.

ANSWER KEY:

(1) $5x + 5$ **(2)** $-7x^2 + 11$ **(3)** $-9y^2 + 2$ **(4)** $2x^2 + 4xy - y + 8$
(5) $-x^2 + 9x - 11y^2$ **(6)** $-4a^2 - 11ab + 16$ **(7)** $-2y^2 - 10y - 6$ **(8)** $2a + 7b - 2c + 1$
(9) $24a^2 - 7ab + 5$ **(10)** $-2y^2 + 6y + 13$

(Challenge) Explanations may vary. One possible answer follows. Yes, if the coefficients of the similar terms are the same, the difference will be a constant, for example $(x - 4) - (x - 8) = 4$. In this case, the answer is a constant.

Name _____ Date _____

WORKSHEET 5.4: SUBTRACTING POLYNOMIALS

Follow the steps below to subtract polynomials:

1. Add the opposite of each term of the polynomial you are subtracting.
2. Simplify by combining similar terms.
3. Add any constants.

EXAMPLES

$(4x - 7) - (3x - 4) = 4x - 7 + (-3x) + 4 = x - 3$

$(3x^2 - 4x + 1) - (-2x^2 + 7x - 9) = 3x^2 - 4x + 1 + 2x^2 + (-7x) + 9 = 5x^2 - 11x + 10$

$(-4a^2 + ab - b) - (-6a^2 - 4ab + 3b) = -4a^2 + ab - b + 6a^2 + 4ab + (-3b) = 2a^2 + 5ab - 4b$

DIRECTIONS: Find the difference.

1. $(7x - 1) - (2x - 6)$
2. $(-3x^2 + 9) - (4x^2 - 2)$
3. $(-3y^2 - 7) - (6y^2 - 9)$
4. $(4x^2 - 3xy - y) - (2x^2 - 7xy - 8)$
5. $(3x^2 + 4x - 10y^2) - (4x^2 - 5x + y^2)$
6. $(-ab + 4) - (4a^2 + 10ab - 12)$
7. $(y^2 - 7) - (3y^2 + 10y - 1)$
8. $(8a - 4c + 10) - (6a - 7b - 2c + 9)$
9. $(12a^2 - 3ab + 4) - (-12a^2 + 4ab - 1)$
10. $(y^2 - y + 1) - (3y^2 - 7y - 12)$

CHALLENGE: If you subtract two polynomials, will your answer ever be a constant? Explain your answer.

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Teaching Notes 5.5: Multiplying Monomials

When students multiply monomials, two errors are common. They may multiply the exponents rather than add them or they may add the coefficients rather than multiply them.

1. Discuss the meaning of bases, exponents, and coefficients. Explain that $3y^2$ means $3 \times y \times y$. y is the base. The exponent 2 means y is a factor 2 times. The coefficient 3 means 3 groups of $y \times y$.
2. Explain that the product of $3y^2 \times 5y^4$ can be written as $3 \times y \times y \times 5 \times y \times y \times y \times y$, which equals $15 \times y^6$ or $15y^6$. Note that the property of exponents for multiplication, $x^m \cdot x^n = x^{m+n}$, is a shortcut for finding this product. Also note the following:
   - If the bases are the same, students can find the product of the bases by adding the exponents.
   - The property of exponents applies only if the bases are the same.
   - Unless otherwise noted, the exponent of a variable is understood to be 1.
   - The coefficients of the monomials are not exponents and should be multiplied.
3. Review the information and examples on the worksheet with your students. Demonstrate how the property of exponents is applied to the problems.

EXTRA HELP:

Placing monomials in parentheses next to each other is another way to show multiplication.

ANSWER KEY:

(1) y^6 **(2)** $14a^{11}$ **(3)** $-18a^6$ **(4)** $21a^3b^3$ **(5)** $-16a^8b^3$ **(6)** $24x^3y^5$ **(7)** $126a^3b^6$ **(8)** $-120a^5b^5$

(Challenge) Margo's answer is correct but there is a flaw in her reasoning. She should have multiplied the coefficients instead of adding them.

Name _____ Date _____

WORKSHEET 5.5: MULTIPLYING MONOMIALS

Use the property of exponents for multiplication, $x^m \cdot x^n = x^{m+n}$, to multiply two or more monomials. Follow the guidelines below:

- Add the exponents if the bases are same.
- Multiply the numerical coefficients.

EXAMPLES

$(2x^3)(4x^7) = 8x^{10}$

$(-3a^5)(6a) = -18a^6$

$(4a^2b^6)(9a^3b^{10}) = 36a^5b^{16}$

DIRECTIONS: Find the product.

1. $(y^2)(y^4)$

2. $(7a^3)(2a^8)$

3. $(-2a^5)(9a)$

4. $(7ab)(3a^2b^2)$

5. $(8a^3b^2)(-2a^5b)$

6. $(-xy^3)(-3xy)(8xy)$

7. $(3b^4)(7ab)(6a^2b)$

8. $(5a^2b^3)(-3a^2b)(8ab)$

CHALLENGE: Margo said that $(2x^2)(2x^2) = 4x^4$. She said that she added the coefficients and added the exponents. Do you agree with her answer and her reasoning? Explain.

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Teaching Notes 5.6: Using Powers of Monomials

Students need to understand the two methods for simplifying the powers of monomials: finding the power of a power or finding the power of a product. Many students confuse the steps of each.

1. Provide these two expressions as examples: $(x^2)^3$ and $(5x^2)^3$. Highlight the following points:
   - Because x^2 is a power, $(x^2)^3$ is a power of a power. The exponent 3 means that x^2 is a factor 3 times. $(x^2)^3$ can be written as $x^2 \cdot x^2 \cdot x^2$ or x^6.
   - Because $5x^2$ is the product of 5 and x^2, $(5x^2)^3$ is the power of a product. The exponent 3 means that $5x^2$ is a factor 3 times. $(5x^2)^3$ can be written as $5x^2 \cdot 5x^2 \cdot 5x^2$ or $125x^6$.
2. Review the properties of integers and powers with your students.
   - A negative integer raised to an even power is positive. $(-2)^2 = 4$
   - A negative integer raised to an odd power is negative. $(-2)^3 = -8$
   - A positive integer raised to any power is positive. $2^3 = 8$
3. Review the information and examples on the worksheet with your students. Be sure to thoroughly discuss the property of exponents for finding the power of a power and for finding the power of a product. If necessary, review 5.5: "Multiplying Monomials."

EXTRA HELP:

Express all powers of numerical coefficients as integers.

ANSWER KEY:

(1) a^8 **(2)** $16a^8$ **(3)** a^{16} **(4)** $225x^{10}$ **(5)** x^3y^3 **(6)** $81x^8$ **(7)** $64a^6$
(8) $-27x^3y^6$ **(9)** $16x^{12}y^8$ **(10)** $16a^2b^2$ **(11)** $16a^2b^2$ **(12)** $1000x^9y^9$

(Challenge) A negative number raised to an odd power is never positive.

Name _____ Date _____

WORKSHEET 5.6: USING POWERS OF MONOMIALS

To simplify the powers of monomials, you must either find the power of a power or find the power of a product. Use the properties of exponents, which are stated below:

- To find the power of a power, the property of exponents states that $(x^m)^n = x^{mn}$.

  $(x^2)^5 = x^{10}$ $(a^2)^3 = a^6$ $(2^3)^3 = (2)^9 = 512$

- To find the power of a product, the property of exponents states that $(xy)^m = x^m y^m$.

  $(3x^5)^2 = 3^2x^{10} = 9x^{10}$ $(-5a^7)^3 = (-5)^3a^{21} = -125a^{21}$ $(-a^2x^4)^4 = (-1)^4(a^8x^{16}) = a^8x^{16}$

DIRECTIONS: Simplify the monomials.

1. $(a^2)^4$ **2.** $(2a^2)^4$ **3.** $(a^8)^2$ **4.** $(15x^5)^2$

5. $(xy)^3$ **6.** $(-3x^2)^4$ **7.** $(2^2a^2)^3$ **8.** $(-3xy^2)^3$

9. $(2x^3y^2)^4$ **10.** $(-4ab)^2$ **11.** $(4ab)^2$ **12.** $(10x^3y^3)^3$

CHALLENGE: Without doing any calculations Tricia said that $(-2xy)^3 = 8x^3y^3$ is incorrect. How was she able to realize this so quickly?

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Teaching Notes 5.7: Multiplying a Polynomial by a Monomial

Multiplying a polynomial by a monomial requires application of the distributive property. Mistakes often occur in use of the distributive property, computation with integers, or application of the property of exponents.

1. Discuss the distributive property, $a(b + c) = ab + ac$, with your students. Provide examples such as $3(8 + 4) = 3 \times 8 + 3 \times 4 = 24 + 12 = 36$. Note that each term in the parentheses is multiplied by 3. Also note that if students follow the order of operations by simplifying the expression inside the parentheses first, 3×12 also equals 36. Depending on the ability of your students, you may wish to review integer operations, particularly subtraction and multiplication.
2. Ask your students to simplify $2a(4a^3 + 6)$ and demonstrate the application of the distributive property. Because the expression inside the parentheses cannot be simplified, students must use the distributive property. Emphasize that each term in the polynomial must be multiplied by the monomial and remind your students that they must now use the property of exponents for multiplication, $x^m \cdot x^n = x^{m+n}$. Thus, $2a(4a^3 + 6) = (2a)(4a^3) + (2a)(6) = 8a^4 + 12a$. Note that they cannot add $8a^4$ and $12a$ because they are not similar terms.
3. Review the information and example on the worksheet with your students. Be sure that your students understand how to apply the distributive property and the property of exponents for multiplication. Discuss the example, explaining the steps in detail.

EXTRA HELP:

Be careful when you multiply the polynomial by a monomial that has a negative coefficient. Apply the rules for multiplying integers.

ANSWER KEY:

(1) $5x + 10$ **(2)** $3x - 12$ **(3)** $4x^2 + 20x$ **(4)** $2x^5 - 14x^2$

(5) $-12a^3 - 3a$ **(6)** $30a^6 - 35a^2$ **(7)** $24x^4y + 32x^3y^2$ **(8)** $7x^5 + 35x^3 - 14x^2$

(9) $6a^3b^2 + 24a^5b^3 + 48a^2b^8$

(Challenge) Khalil is partially correct. His work is right but his answer is not in simplest form. It should be $7x^2 - 14x$.

Name _____ Date _____

WORKSHEET 5.7: MULTIPLYING A POLYNOMIAL BY A MONOMIAL

Follow the steps below to multiply a polynomial by a monomial:

1. Multiply each term of the polynomial by the monomial. (Use the distributive property, $a(b + c) = ab + ac$, and multiply the monomials using the property of exponents for multiplication, $x^m + x^n = x^{m+n}$.)
2. Find the sum.

EXAMPLE

$$-8x^4(2x^2 - 7x + 5) = (-8x^4)(2x^2) + (-8x^4)(-7x) + (-8x^4)(5) = -16x^6 + 56x^5 - 40x^4$$

DIRECTIONS: Find the product.

1. $5(x + 2)$
2. $3(x - 4)$
3. $4x(x + 5)$
4. $2x^2(x^3 - 7)$
5. $-3a(4a^2 + 1)$
6. $5a^2(6a^4 - 7)$
7. $8x^2y(3x^2 + 4xy)$
8. $7x^2(x^3 + 5x - 2)$
9. $6a^2b^2(a + 4a^3b + 8b^2)$

CHALLENGE: Khalil said that $3x(x + 2) + 4x(x - 5) = 3x^2 + 6x + 4x^2 - 20x$. Is he correct? Explain.

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Teaching Notes 5.8: Multiplying Two Binomials

To multiply two binomials, students may use the distributive property two times. Because this can be confusing, the acronym FOIL is helpful for remembering the process.

1. Explain the steps for multiplying two binomials by applying the distributive property twice.
   - Start with the first term of the first binomial. Multiply each term of the second binomial by the first term of the first binomial.
   - Multiply each term of the second binomial by the second term of the first binomial.
   - Simplify the answer.
2. Provide this example to demonstrate the application of the distributive property:

$$(x + 2)(x + 7) = x(x + 7) + 2(x + 7) = x^2 + 7x + 2x + 14 = x^2 + 9x + 14$$

3. Explain that FOIL is an acronym that refers to the *first, outer, inner*, and *last* terms of binomials. This will help your students remember how to multiply binomials.
4. Review the information and examples on the worksheet with your students. Be sure that your students can apply FOIL. Also remind them that if any terms in the binomials are subtracted, they must rewrite the subtraction as addition. Review the steps for rewriting in detail if necessary.

EXTRA HELP:

The FOIL method applies to only the multiplication of two binomials.

ANSWER KEY:

(1) $x^2 + 2x - 15$ **(2)** $x^2 + x - 56$ **(3)** $x^2 - 7x + 6$ **(4)** $8x^2 - 2x - 3$

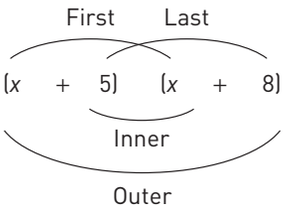
(5) $-2x^2 + 19x - 24$ **(6)** $x^3 + 3x^2 + x + 3$ **(7)** $x^3 + 2x^2 - 5x - 10$ **(8)** $2x^2 - 5xy - 3y^2$

(Challenge) Subtraction was not rewritten as addition. $(x - 3)(x + 4) = (x + (-3))(x + 4) = x^2 + 4x + (-3x) + (-12) = x^2 + x - 12$

Name _____ Date _____

WORKSHEET 5.8: MULTIPLYING TWO BINOMIALS

FOIL is an acronym that stands for *first*, *outer*, *inner*, *last*. It can help you to multiply two binomials. The diagram below illustrates how to use FOIL to multiply $(x + 5)(x + 8)$.



- The first terms of each binomial are x and x.
- The outer terms are x and 8.
- The inner terms are 5 and x.
- The last terms are 5 and 8.

After you have identified the correct terms, find the product of each pair. Then find the sum.

EXAMPLE

$$(x + 5)(x + 8) = x^2 + 8x + 5x + 40 = x^2 + 13x + 40$$

If any terms in the binomials are subtracted, rewrite the subtraction as addition and add the opposite. Find the product of each pair, then find the sum.

EXAMPLE

$$(x - 1)(x + 7) = (x + (-1))(x + 7) = x^2 + 7x + (-1)(x) + (-1)(7) = x^2 + 6x - 7$$

DIRECTIONS: Find the product.

1. $(x - 3)(x + 5)$

2. $(x + 8)(x - 7)$

3. $(x - 1)(x - 6)$

4. $(2x + 1)(4x - 3)$

5. $(-2x + 3)(x - 8)$

6. $(x^2 + 1)(x + 3)$

7. $(x + 2)(x^2 - 5)$

8. $(x - 3y)(2x + y)$

CHALLENGE: Find and correct the error in the following:

$$(x - 3)(x + 4) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12$$

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Teaching Notes 5.9: Multiplying Two Polynomials

Multiplying polynomials involves using the distributive property and simplifying similar terms. Common errors include incorrectly applying the distributive property, incorrectly multiplying monomials, and incorrectly combining similar terms.

1. Explain that when students multiply polynomials other than two binomials, they must apply the distributive property, multiply monomials, and then combine similar terms. You may mention that when multiplying two binomials, they may use the method described here or use the steps of the acronym FOIL. Explain that FOIL is a mnemonic aid that summarizes how to apply the distributive property when students multiply two binomials.
2. Review the information and example on the worksheet with your students. Because of the complexity of the steps for multiplying polynomials, you might find it helpful to discuss them in detail. Go over the example thoroughly and emphasize the steps of the distributive property. Especially note that the second term of the first polynomial is -4 and each term of the second polynomial is multiplied by -4.

EXTRA HELP:

Because several skills and concepts are applied when multiplying two polynomials, work carefully to avoid careless mistakes. Always double-check your work.

ANSWER KEY:

(1) $8x^3 - 2x^2 + x + 2$ **(2)** $x^3 + 5x^2 + 7x + 2$ **(3)** $x^3 - 4x^2 + 4x - 3$

(4) $2x^3 - x^2 - 7x + 20$ **(5)** $3x^3 - 14x^2 + 17x - 6$ **(6)** $2x^4 - 6x^3 - 7x^2 + 3x + 3$

(7) $x^4 - 3x^3 + 3x - 1$ **(8)** $x^5 - 7x^4 + 2x^3 - 4x^2 + 28x - 8$ **(9)** $x^4 + 7x^3 - 9x^2 + x$

(Challenge) Billy is incorrect. He multiplied each term of the second polynomial by 3 instead of by -3. The correct answer is $x^3 - 4x^2 + 2x + 3$.

Name _____ Date _____

WORKSHEET 5.9: MULTIPLYING TWO POLYNOMIALS

Follow the steps below to multiply two polynomials:

1. Use the distributive property to find the product of the first term of the first polynomial and every term of the second polynomial.
2. Use the distributive property to write the product of the second term of the first polynomial and every term of the second polynomial. Continue this process until all the products of all the terms of the first polynomial are written.
3. Combine similar terms.

EXAMPLE

- Multiply $(3x^2 - 4)(2x^2 + 5x - 1)$.
- $3x^2$ is the first term of the first polynomial. Write the product of $3x^2$ and every term of the second polynomial. $3x^2(2x^2 + 5x - 1) = 6x^4 + 15x^3 - 3x^2$
- -4 is the second term of the first polynomial. Write the product of -4 and every term of the second polynomial. $-4(2x^2 + 5x - 1) = -8x^2 - 20x + 4$
- Combine similar terms. $6x^4 + 15x^3 - 3x^2 - 8x^2 - 20x + 4 = 6x^4 + 15x^3 - 11x^2 - 20x + 4$

DIRECTIONS: Multiply.

1. $(2x + 1)(4x^2 - 3x + 2)$

2. $(x + 2)(x^2 + 3x + 1)$

3. $(x - 3)(x^2 - x + 1)$

4. $(2x + 5)(x^2 - 3x + 4)$

5. $(3x - 2)(x^2 - 4x + 3)$

6. $(2x^2 - 1)(x^2 - 3x - 3)$

7. $(x^2 - 1)(x^2 - 3x + 1)$

8. $(x^3 - 4)(x^2 - 7x + 2)$

9. $(x^2 - x)(x^2 + 8x - 1)$

CHALLENGE: Billy multiplied $(x - 3)(x^2 - x - 1)$ and said that the product was $x^3 + 2x^2 - 4x - 3$. Is he correct? If you disagree with his answer, explain his error and provide the correct answer.

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Teaching Notes 5.10: Dividing Monomials

The process of dividing monomials involves subtracting the exponents of common bases. Confusion arises when students wonder why they are subtracting when they are supposed to be dividing.

1. Review the property of exponents for multiplication, $x^m \cdot x^n = x^{m+n}$. Present examples such as $3x^2 \cdot 4x^8$. Note that the product $12x^{10}$ was found by multiplying the coefficients of the terms and adding the exponents of the common bases. Provide more examples if necessary.
2. Explain that division is the inverse, or opposite, operation of multiplication. When dividing monomials, instead of multiplying the coefficients as in multiplication, students divide or simplify the coefficients. Instead of adding the exponents as in multiplication, they subtract the exponents when dividing. Introduce the property of exponents for division, $\frac{x^m}{x^n} = x^{m-n}$ where $x \neq 0$.
3. Explain that all powers of powers must be simplified before monomials can be divided. Provide examples such as $\frac{(2x^2)^3}{3x}$, which must be simplified to $\frac{8x^6}{3x}$ before dividing. You might also find it helpful to review 5.6: "Using Powers of Monomials."
4. Review the information and example on the worksheet with your students.

EXTRA HELP:

Check your work by multiplying the quotient by the divisor. The answer should equal the monomial in the numerator.

ANSWER KEY:

(1) $2x$ **(2)** $7x$ **(3)** $250a^5$ **(4)** 1 **(5)** z^2 **(6)** $3a^7$

(Challenge) Rebecca's answer is incorrect. She should have subtracted the exponents rather than divided them. $\frac{18x^4y^{10}}{9x^4y^5} = 2y^5$

Name _____ Date _____

WORKSHEET 5.10: DIVIDING MONOMIALS

Use the property of exponents for division, $\frac{x^m}{x^n} = x^{m-n}$, where $x \neq 0$ to divide monomials. Follow these steps:

1. Simplify powers of powers, if necessary.
2. Simplify the quotients of the coefficients by dividing.
3. Identify common bases. Divide by subtracting the exponent in the denominator from the exponent in the numerator of each common base. If the exponents are the same, the quotient of those common bases is equal to 1.

EXAMPLE

Divide $\frac{(2x^3)^3}{2x^7}$.

Simplify powers of powers. $\frac{(2x^3)^3}{2x^7} = \frac{8x^9}{2x^7}$

Simplify the coefficients. $\frac{8x^9}{2x^7} = 4\frac{x^9}{x^7}$

Divide common bases by subtracting the exponents. $4\frac{x^9}{x^7} = 4x^2$

DIRECTIONS: Divide.

1. $\frac{6x^2}{3x}$
2. $\frac{21x^5}{3x^4}$
3. $\frac{(10a^2)^3}{4a}$
4. $\frac{8a^3b^3}{(2ab)^3}$
5. $\frac{x^2y^4z^3}{x^2y^4z}$
6. $\frac{15a^8}{5a}$

CHALLENGE: Rebecca found that $\frac{18x^4y^{10}}{9x^4y^5} = 2xy^2$. Is she correct? If not, what did she do wrong and what is the correct answer?

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Teaching Notes 5.11: Dividing Polynomials

The steps for dividing one polynomial by another are similar to the steps for dividing whole numbers. Just as with long division, students often become confused by the process or make errors in computation.

1. Explain that polynomials can be divided following the same process as dividing whole numbers: divide, multiply, subtract, and bring down the next term. To demonstrate these steps, offer an example by using long division to divide 328 by 15. The quotient is 21, r 13, or $21\frac{13}{15}$.
2. Explain that before dividing polynomials, each polynomial must be written so that the exponents are in descending order. Provide this example: $(x - 1 + 2x^2) \div (3 + x)$ must be rewritten as $(2x^2 + x - 1) \div (x + 3)$. If an exponent is "missing," the term must be rewritten with 0 as the coefficient. For example, $(x^3 + 2x - 1) \div (x + 4)$ is written in descending order but there is no x^2 term in the first polynomial. This polynomial must be rewritten as $x^3 + 0x^2 + 2x - 1$.
3. Review the information and example on the worksheet with your students. Discuss each step for division. Caution your students to be especially careful when subtracting. Depending on the abilities of your students, you might find it helpful to review 5.4: "Subtracting Polynomials."

EXTRA HELP:

You may check your answer by multiplying the quotient by the divisor and adding the numerator of the fraction that remains.

ANSWER KEY:

(1) $x - 1 + \frac{5}{x+4}$ **(2)** $2x - 4 + \frac{9}{x+1}$ **(3)** $3x - 10 + \frac{10}{x+2}$ **(4)** $x + 2 + \frac{9}{x-4}$

(5) $2x - 1 + \frac{1}{2x-1}$ **(6)** $x - 2 + \frac{3}{x+2}$

(Challenge) Tyrell did not write the terms in descending order of the exponents. He should have included an x term. The correct way of rewriting the polynomial is $x^3 + x^2 + 0x + 1$.

Name _____ Date _____

WORKSHEET 5.11: DIVIDING POLYNOMIALS

Dividing polynomials is similar to division of whole numbers. Follow the steps below:

1. Divide.
2. Multiply.
3. Subtract.
4. Bring down the next term and repeat the process. If there is no term to bring down, write the remainder over the divisor and add it to the quotient.

EXAMPLE

Divide $(2x^2 + x - 1)$ by $(x + 3)$.

Divide $2x^2$ by x. Place $2x$ in the quotient.

Multiply $2x$ by $(x + 3)$.

Subtract $(2x^2 + 6x)$ from $(2x^2 + x)$.

Bring down -1.

Divide $-5x$ by x. Place -5 in the quotient.

Multiply -5 by $(x + 3)$.

Subtract $(-5x - 15)$ from $(-5x - 1)$.

$$
\begin{array}{r}
2x - 5 \quad + \dfrac{14}{x+3} \\
x+3\,\overline{)\,2x^2 + \ x - 1} \\
\underline{2x^2 + 6x} \\
-5x - 1 \\
\underline{-5x - 15} \\
14
\end{array}
$$

There are no terms to bring down. Write the remainder over the divisor and add it to the quotient. $2x - 5 + \dfrac{14}{x+3}$

DIRECTIONS: Divide.

1. $(x^2 + 3x + 1) \div (x + 4)$
2. $(2x^2 - 2x + 5) \div (x + 1)$
3. $(3x^2 - 4x - 10) \div (x + 2)$
4. $(x^2 - 2x + 1) \div (x - 4)$
5. $(-4x + 4x^2 + 2) \div (2x - 1)$
6. $(x^2 - 1) \div (x + 2)$

CHALLENGE: Tyrell tried to divide $(x^2 + x^3 + 1)$ by $(x + 7)$ but the terms did not line up. Explain why this happened.

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Teaching Notes 5.12: Finding the Greatest Common Factor of Two or More Monomials

Several steps are needed to find the greatest common factor of two or more monomials. A common mistake students make is to choose the greatest exponent of a common base.

1. Explain that the greatest common factor of two or more monomials is the common factor that has the greatest coefficient and the greatest degree in each variable. For example, $6xy^2$ is the greatest common factor of $12xy^3$ and $18x^2y^2$. 6 is the greatest common factor of the coefficients 12 and 18, and xy^2 is the greatest common factor of xy^3 and x^2y^2.
2. Review the process of finding the greatest common factor of two or more numbers. Students will use this skill to find the greatest common factor of the coefficients of each term.
3. Explain that students must select the smaller exponent of each common factor and each common variable to find the greatest common factor. For example, if the common factors are x^2 and x^3, they must choose x^2 because x^2 is a factor of x^3. If necessary, express x^2 as $x \cdot x$ and x^3 as $x \cdot x \cdot x$ to help your students realize that x^2 is the greatest common factor.
4. Review the information and example on the worksheet with your students. Work through the example as a class, noting each step.

EXTRA HELP:

If two or more monomials have no terms in common, the greatest common factor is 1.

ANSWER KEY:

(1) x **(2)** $7ab^2$ **(3)** $2x$ **(4)** ab **(5)** a^2 **(6)** $12x^2y^2$ **(7)** 13 **(8)** $8xyz$

(Challenge) Sue is incorrect. She should have first simplified $(2xy)^2$, which is $4x^2y^2$, and then used this term to find the greatest common factor. The greatest common factor is x^2y.

Name _____ Date _____

WORKSHEET 5.12: FINDING THE GREATEST COMMON FACTOR OF TWO OR MORE MONOMIALS

The greatest common factor is the largest factor that two or more monomials have in common. To find the greatest common factor of two monomials, follow the steps below:

1. Use exponents to write the prime factorization of each coefficient.
2. Find the prime numbers that are common factors of each coefficient.
3. Choose the smaller exponent of each common prime number and the smaller exponent of each common variable.
4. Find the product of the common factors to find the greatest common factor.

EXAMPLE

Find the greatest common factor of $20a^2bc^3$ and $30a^5b^2$.

$$20a^2bc^3 = 2^2 \times 5 \times a^2 \times b \times c^3 \qquad 30a^5b^2 = 2 \times 3 \times 5 \times a^5 \times b^2$$

Common factors are 2, 5, a, and b.
1 is the smallest exponent of 2, 5, and b.
2 is the smaller exponent of a.
The greatest common factor of $20a^2bc^3$ and $30a^5b^2$ is $2 \times 5 \times a^2 \times b$ or $10a^2b$.

DIRECTIONS: Find the greatest common factor of each pair of monomials.

1. $3x^2, 2x$
2. $7a^2b^2, 14ab^3c$
3. $12xy, 10x$
4. $15ab, 7a^2b^2$
5. $9a^2, 16a^3$
6. $36x^2y^2, 12x^2y^2$
7. $13a, 26b^2$
8. $32xyz^4, 8xy^2z$

CHALLENGE: Sue tried to find the greatest common factor of x^2y and $(2xy)^2$. She looked at the common variables x and y and used the smaller exponent of both variables. She concluded that the greatest common factor is xy. Is she correct? Explain your reasoning.

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Teaching Notes 5.13: Factoring Polynomials by Finding the Greatest Monomial Factor

Factoring a polynomial by using the greatest monomial factor is difficult for many students. They may fail to find the greatest monomial factor or they may make a mistake in division.

1. Explain that although the term "greatest common factor" applies to numbers and monomials, the greatest monomial factor applies to polynomials. It is the greatest common monomial factor of each of the terms of the polynomial.
2. Provide examples of the greatest monomial factor such as the following:
   - $4x^2 + 7x$; x is the greatest monomial factor.
   - $12a^2 + 16ab$; $4a$ is the greatest monomial factor.
   - $3x^2 + 2b$; 1 is the greatest monomial factor.
3. Explain that because the greatest monomial factor of a polynomial is a factor of each term, the other factors can be found by dividing the polynomial by the greatest monomial factor. Using the first example above, divide each term of the polynomial by x. $\frac{4x^2}{x} + \frac{7x}{x} = 4x + 7$ Thus, the factors of $4x^2 + 7x$ are x and $(4x+ 7)$. Using the same procedure in the second example, the factors of $12a^2 + 16ab$ are $4a$ and $(3a+ 4b)$. Explain that $3x^2 + 2b$ cannot be factored because 1 is the greatest monomial factor.
4. Review the information and example on the worksheet with your students. If necessary, discuss in detail and demonstrate the steps for finding the greatest monomial factor and factoring the example.

EXTRA HELP:

Be sure you have found the greatest monomial factor before you divide.

ANSWER KEY:

(1) $3(x + 5)$ **(2)** Cannot be factored **(3)** $ab(8 - 3b)$ **(4)** $10x(2xy - 1 + 4x^2)$
(5) $x(49x - 1)$ **(6)** $5x^2y^3(7x^2 - 3y^2)$ **(7)** $a^2b(1 + 2ab)$ **(8)** $6x(2y - x)$

(Challenge) Eva is incorrect. $2xy$ is not the greatest monomial factor; $2xy^2$ is. The correct factorization is $2xy^2(2x - 5)$.

Name _____ Date _____

WORKSHEET 5.13: FACTORING POLYNOMIALS BY FINDING THE GREATEST MONOMIAL FACTOR

To factor a polynomial by finding its greatest monomial factor, follow the steps below:

1. Find the greatest monomial factor of the polynomial.
2. Divide each term of the polynomial by the greatest monomial factor. (If 1 is the greatest monomial factor, the polynomial cannot be factored using this strategy.)
3. Write the product of the greatest monomial factor and the polynomial found in step 2.
4. Use the distributive property to check your work.

EXAMPLE

Factor $2x^3 - 6x^2 + 10x$.

The greatest monomial factor of the polynomial is $2x$.

$$\frac{2x^3}{2x} - \left(\frac{6x^2}{2x}\right) + \frac{10x}{2x} = x^2 - 3x + 5$$

$$2x(x^2 - 3x + 5) = 2x^3 - 6x^2 + 10x$$

DIRECTIONS: Factor each polynomial, if possible. If the polynomial cannot be factored, write "cannot be factored."

1. $3x + 15$
2. $x^2 + 4y$
3. $8ab - 3ab^2$
4. $20x^2y - 10x + 40x^3$
5. $49x^2 - x$
6. $35x^4y^3 - 15x^2y^5$
7. $a^2b + 2a^3b^2$
8. $12xy - 6x^2$

CHALLENGE: Eva factored $4x^2y^2 - 10xy^2$ as $2xy(2xy - 5y)$. Is she correct? If not, explain why not and provide the correct answer.

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Teaching Notes 5.14: Factoring the Difference of Squares

Factoring the difference of squares is a skill that requires students to identify the square of a number and write the square of a variable expression. Both steps can be problem areas.

1. Explain that a binomial of the form $a^2 - b^2$ is called the "difference of squares" because it is the difference of two quantities, each raised to the second power.
2. Present some examples of square numbers such as 1, 4, 9, 16 . . . , noting that each square number can be expressed as a number raised to the second power. $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, and so on.
3. Explain that variable expressions raised to an even power can also be expressed as the square of the base by dividing the exponent by 2. To illustrate this, present examples such as $y^4 = (y^2)^2$, $x^6 = (x^3)^2$, and $a^2b^{10} = (ab^5)^2$.
4. Review the information and example on the worksheet with your students. Make sure that your students understand how to use the formula $a^2 - b^2 = (a + b)(a - b)$.

EXTRA HELP:

There is no formula for factoring the sum of two squares.

ANSWER KEY:

(1) $(x - 10)(x + 10)$ **(2)** $(1 - 4x^5)(1 + 4x^5)$ **(3)** $(7 - y^6)(7 + y^6)$ **(4)** $(3 - 10x)(3 + 10x)$
(5) $(8 - y)(8 + y)$ **(6)** $(xy - 5)(xy + 5)$ **(7)** $(12 - x^4)(12 + x^4)$ **(8)** $(x^3 - 11)(x^3 + 11)$
(9) $(a^2b - 9)(a^2b + 9)$ **(10)** $(x^3y^3 - 12)(x^3y^3 + 12)$ **(11)** $(2x + 1)(2x - 1)$ **(12)** $(6x^5 - 5y)(6x^5 + 5y)$

(Challenge) Crystal is incorrect. $4 + x^{16}$ is the sum of two squares, which cannot be factored using the formula for factoring the difference of squares.

Name _____ Date _____

WORKSHEET 5.14: FACTORING THE DIFFERENCE OF SQUARES

Follow the steps below to factor the difference of squares:

1. Determine if the binomial can be written as the difference of two numbers or expressions raised to the second power. Write each as a base squared.
2. Substitute each base in the formula $a^2 - b^2 = (a + b)(a - b)$.
3. Use FOIL to check your work.

EXAMPLE

Factor $25 - 9x^8$.

$25 = 5^2$; $9x^8 = (3x^4)^2$

$a = 5$; $b = (3x^4)$; $5^2 - (3x^4)^2 = (5 + 3x^4)(5 - 3x^4)$

$(5 + 3x^4)(5 - 3x^4) = 25 - 15x^4 + 15x^4 - 9x^8 = 25 - 9x^8$

DIRECTIONS: Factor each binomial.

1. $x^2 - 100$
2. $1 - 16x^{10}$
3. $49 - y^{12}$
4. $9 - 100x^2$
5. $64 - y^2$
6. $x^2y^2 - 25$
7. $144 - x^8$
8. $x^6 - 121$
9. $a^4b^2 - 81$
10. $x^6y^6 - 144$
11. $4x^2 - 1$
12. $36x^{10} - 25y^2$

CHALLENGE: Crystal factored $4 + x^{16}$ as $(2 - x^8)(2 + x^8)$. Was she correct? Explain your reasoning.

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Teaching Notes 5.15: Factoring Trinomials if the Last Term Is Positive

Factoring a trinomial means to write the trinomial as the product of its factors. To factor a trinomial in the form $x^2 + bx + c$, you must find two factors of c whose sum equals b. Because c is a positive term, it is necessary to understand that the factors of c must both be positive or both be negative. Students often forget this fact, which leads to mistakes.

1. Explain that a trinomial may be the product of two binomials. For example, $x^2 + 5x + 4$ is the product of $(x + 4)$ and $(x + 1)$. When students factor a trinomial, they must find two binomials whose product is equal to the trinomial. If the last term, c, of the trinomial is a positive integer, finding the factors of c is necessary to factor the trinomial. (c may be negative but that will be discussed in 5.16: "Factoring Trinomials if the Last Term Is Negative.")
2. Review the information and example on the worksheet with your students, particularly the steps for factoring trinomials. When discussing the examples, emphasize that when the last term is positive, the factors are a pair of positive numbers or a pair of negative numbers. The sign of the second term, b, indicates whether the pair is positive or negative.
3. Point out that not all trinomials can be factored. They may be factored only when the sum of a pair of factors of c is equal to b.

EXTRA HELP:

When b is positive, the factors of c will also be positive. When b is negative, the factors of c will be negative.

ANSWER KEY:

(1) $(x + 1)(x + 5)$	**(2)** $(x + 2)(x + 4)$	**(3)** $(x - 3(x - 2)$	**(4)** $(x - 4)(x - 5)$
(5) $(x + 5)(x + 2)$	**(6)** $(x + 1)(x + 22)$	**(7)** Cannot be factored	**(8)** $(x - 4)(x - 3)$
(9) $(x + 5)(x + 3)$	**(10)** $(x - 16)(x - 1)$	**(11)** Cannot be factored	**(12)** $(x - 8)(x - 5)$

(Challenge) Kim could know the signs of the factors without knowing the values of b and c by using this reasoning: if b is negative, both factors of c must be negative. If b is positive, both factors of c must be positive.

Name _____ Date _____

WORKSHEET 5.15: FACTORING TRINOMIALS IF THE LAST TERM IS POSITIVE

When factoring trinomials in the form $x^2 + bx + c$, you must find two factors of c whose sum equals b. Follow the steps below to factor trinomials if the last term is positive:

1. List all the pairs of factors of c.
2. Consider b, the coefficient of x.
   - If b is positive, choose the two positive factors of c whose sum equals b.
   - If b is negative, choose the two negative factors of c whose sum equals b.
3. Place these factors into factored form.
4. Check by multiplying the factors.

EXAMPLE

Factor $x^2 + 7x + 10$.

The factors of 10: 1 and 10; 2 and 5; -1 and -10; -2 and -5.

Because $b = 7$, choose 2 and 5 as the factors of c.

$$(x + 2)(x + 5)$$

Check: $(x + 2)(x + 5) = x^2 + 5x + 2x + 10 = x^2 + 7x + 10$

DIRECTIONS: Factor each trinomial. If the trinomial cannot be factored, write "cannot be factored."

1. $x^2 + 6x + 5$
2. $x^2 + 6x + 8$
3. $x^2 - 5x + 6$
4. $x^2 - 9x + 20$
5. $x^2 + 7x + 10$
6. $x^2 + 23x + 22$
7. $x^2 - 10x + 30$
8. $x^2 - 7x + 12$
9. $x^2 + 8x + 15$
10. $x^2 - 17x + 16$
11. $x^2 + 8x + 12$
12. $x^2 - 13x + 40$

CHALLENGE: Kim told James that if c is positive and he told her the sign of b, she would tell him the signs of their factors. How could she know this without knowing the actual value of b?

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Teaching Notes 5.16: Factoring Trinomials if the Last Term Is Negative

To factor a trinomial in the form $x^2 + bx + c$, students must determine the two factors of the negative number c whose sum equals b. Once they realize that this product can be found only by multiplying a positive integer and a negative integer, they must select the correct pair of integers.

1. Discuss the procedures for factoring a trinomial. Students should understand that they need to find two numbers whose product is c and whose sum is b. For example, in $x^2 + 3x - 10$, $c = -10$ and $b = 3$. Students must find two factors of -10 whose sum is 3. 5 and -2 are the factors.
2. Emphasize that when the last term is negative, one factor will be positive and the other will be negative. Remind your students that a product of two integers is negative when the signs of its factors are different.
3. Review the information and example on the worksheet with your students. Discuss the steps for factoring the example. Be sure to note that either of two pairs of factors could be used to factor the trinomial. Emphasize that students must select two factors whose sum is 2. Depending on their abilities, you may find it helpful to review adding integers.
4. Explain that not all trinomials can be factored. They can be factored only when the sum of a pair of factors of c is equal to b.

EXTRA HELP:

Even though the product of two integers may be negative, the sum could be positive.

ANSWER KEY:

(1) $(x - 2)(x + 3)$	**(2)** $(x + 4)(x - 2)$	**(3)** $(x - 1)(x + 5)$	**(4)** $(x - 7)(x + 2)$
(5) $(x + 8)(x - 3)$	**(6)** $(x - 1)(x + 2)$	**(7)** $(x - 10)(x + 4)$	**(8)** $(x - 5)(x + 6)$
(9) Cannot be factored	**(10)** $(x - 4)(x + 3)$	**(11)** $(x - 8)(x + 4)$	**(12)** $(x + 5)(x - 2)$

(Challenge) Kelli is incorrect. There are six pairs of factors of -20 and she must have overlooked two pairs. One pair of factors whose sum is -19 is -20 and 1. The polynomial can be factored as $(x - 20)(x + 1)$.

Name _____ Date _____

WORKSHEET 5.16: FACTORING TRINOMIALS IF THE LAST TERM IS NEGATIVE

In order to factor trinomials in the form $x^2 + bx + c$, you must find two integers whose product is c and whose sum is b. Follow the steps below to factor trinomials if the last term is negative:

1. List all pairs of the factors of c. Because c is negative, one factor in each pair will be positive and one will be negative.
2. Choose two factors of c that add up to b.
3. Place these factors into factored form.
4. Check by multiplying the factors.

EXAMPLE

Factor $x^2 + 2x - 3$.

The factors of -3: -3 and 1; 3 and -1.

Because $b = 2$, choose 3 and -1 as the factors of c.

$$(x + 3)(x - 1)$$

Check: $(x + 3)(x - 1) = x^2 - x + 3x - 3 = x^2 + 2x - 3$

DIRECTIONS: Factor each trinomial. If the trinomial cannot be factored, write "cannot be factored."

1. $x^2 + x - 6$
2. $x^2 + 2x - 8$
3. $x^2 + 4x - 5$
4. $x^2 - 5x - 14$
5. $x^2 + 5x - 24$
6. $x^2 + x - 2$
7. $x^2 - 6x - 40$
8. $x^2 + x - 30$
9. $x^2 - 5x - 15$
10. $x^2 - x - 12$
11. $x^2 - 4x - 32$
12. $x^2 + 3x - 10$

CHALLENGE: Kelli tried to factor $x^2 - 19x - 20$ and determined that the polynomial could not be factored. She found four pairs of factors of -20, but none added up to -19. Is she correct? Explain your reasoning.

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Teaching Notes 5.17: Factoring by Grouping

Factoring by grouping involves finding and factoring the greatest monomial factor of two binomials and writing the expression as the product of binomials. An error in any part of this process will result in an incorrect factorization.

1. Explain that factoring by grouping may be used to factor polynomials that have four terms, provided that each group of two terms has a common monomial factor. Depending on the abilities of your students, you may wish to review 5.13: "Factoring Polynomials by Finding the Greatest Monomial Factor."
2. Explain that factoring by grouping requires students to rearrange the terms into two groups and then find the greatest monomial factor of each group. A binomial that is a common factor of both groups will result. Note that if a common binomial factor does not result, students may have to regroup the terms or the polynomial cannot be factored by grouping.
3. Review the information and example on the worksheet with your students. Note that the example is factored in two different ways with the same result. Also note that each group of two terms has a common monomial factor. Students cannot rewrite the example as $(xy + 10) + (2x + 5y)$ because neither $xy + 10$ nor $2x + 5y$ has a common factor. Explain that if the binomial is a common factor of the expression, it should be factored out. Remind your students that when checking the answer in the example, xy is the same as yx, and similar terms can be added in any order.

EXTRA HELP:

A polynomial can be factored only by grouping if it has a common binomial factor.

ANSWER KEY:

(1) $(x - 2)(y + 3)$ **(2)** $(y + 2)(x - 3)$ **(3)** $(y - 1)(3x + 2)$ **(4)** $(x + 3)(y - 7)$ **(5)** $(x + 3)(y - 4)$
(6) $(x^2 + 2)(y + 2)$ **(7)** $(x - 2y)(x - 4)$ **(8)** $(xy - 3)(x - 4)$ **(9)** $(2x - 1)(y - 3)$

(Challenge) The teacher is correct. Had Miguel regrouped the terms as $(2x^2 - x) - (20x + 10)$ and wrote $x(2x - 1) - 10(2x - 1)$, he would have found the correct factorization, which is $(x - 10)(2x - 1)$.

Name _____ Date _____

WORKSHEET 5.17: FACTORING BY GROUPING

Some polynomials that have four terms may be factored by grouping terms that have common factors. Follow the steps below:

1. Group the four terms into two groups so that each group has a common monomial factor.
2. Factor the greatest monomial factor of each binomial. A binomial that is a common factor should result.
3. Factor out the binomial that is a common factor.
4. Check by multiplying the factors.

EXAMPLE

Factor $xy + 2x + 5y + 10$ by grouping.

$(xy+ 2x) + (5y+ 10)$	or	$(xy+ 5y) + (2x+ 10)$
$x(y + 2) + 5(y + 2)$	or	$y(x + 5) + 2(x + 5)$
$(y + 2)(x + 5)$	or	$(x + 5)(y + 2)$
$(y + 2)(x + 5) + yx + 5y + 2x + 10$	or	$(x + 5)(y + 2) + xy + 2x + 5y + 10$

DIRECTIONS: Factor by grouping.

1. $xy + 3x - 2y - 6$
2. $xy + 2x - 3y - 6$
3. $3xy - 2 + 2y - 3x$
4. $xy - 7x + 3y - 21$
5. $xy - 4x + 3y - 12$
6. $x^2y + 2y + 2x^2 + 4$
7. $x^2 - 4x - 2xy + 8y$
8. $x^2y - 4xy - 3x + 12$
9. $2xy - 6x - y + 3$

CHALLENGE: Miguel tried to factor $2x^2 - x - 20x + 10$ by grouping. He rewrote the polynomial as $(2x^2 + 10) - (x + 20)$ and factored $2(x^2 + 5) - (x + 20)$. He said that the binomials in parentheses are different and therefore could not be factored by grouping. His teacher said that the polynomial could be factored by grouping. Who is correct? Explain your answer.

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Teaching Notes 5.18: Factoring Trinomials if the Leading Coefficient Is an Integer Greater Than 1

Factoring trinomials when the leading coefficient is an integer greater than 1 requires finding factors of the product, finding the appropriate sum, and then factoring by grouping. This can prove to be a difficult task for students because students must find all of the factors.

1. Review factoring trinomials by presenting this example: $x^2 + 5x + 4$. Note that students must find the factors of 4 whose sum equals 5. $x^2 + 5x + 4 = (x + 4)(x + 1)$
2. Contrast this with factoring $2x^2 - 11x + 12$. Note that the leading coefficient is 2. To factor this trinomial, students must find the product of both the coefficient of x^2 and the third term of the trinomial. This product is 2×12 or 24. Students must identify two factors of 24 whose sum is -11 and then factor by grouping. $2x^2 - 11x + 12 = (2x - 3)(x - 4)$
3. Review the information and example on the worksheet with your students. Depending on their abilities, you may find it helpful to review 5.17: "Factoring by Grouping." Emphasize that your students must find all of the factors of *ac*, including both the positive and negative factors. From this list of factors, they should find a pair of factors whose sum is the middle term. Remind them to always check their answers.

EXTRA HELP:

When you write the expression, you may have to rearrange terms so that there is a common monomial factor.

ANSWER KEY:

(1) $(2x+ 3)(x+ 2)$ **(2)** $(3x - 1)(x + 5)$ **(3)** Cannot be factored **(4)** $(2x - 1)(4x + 3)$
(5) $(3x - 1)(2x + 3)$

(Challenge) Ronnie must select a pair of factors of -6 whose sum is -1, in this case -3 and 2.
$6x^2 - x - 1 = 6x^2 + 2x - 3x - 1 = 2x(3x + 1) - (3x + 1) = (2x - 1)(3x + 1)$

Name _____ Date _____

WORKSHEET 5.18: FACTORING TRINOMIALS IF THE LEADING COEFFICIENT IS AN INTEGER GREATER THAN 1

Follow the steps below to factor trinomials written as $ax^2 + bx + c$ when the leading coefficient is an integer greater than 1:

1. Find the product of a and c. List the factors of the product.
2. Find a pair of factors whose sum is b. (If the sum cannot be found, the trinomial cannot be factored.)
3. Express bx as the sum found in step 2 and substitute these values in the trinomial.
4. Factor by grouping.
5. Check by multiplying the factors.

EXAMPLE

Factor $5x^2 - 18x - 8$.
$ac = -40$. The pairs of factors of -40 are 1, -40; -1, 40; 2, -20; -2, 20; 4, -10; -4, 10; 5, -8; -5, 8.
Because $b = -18$, select 2 and -20 because $2 + (-20) = -18$.
$bx = 2x - 20x$. Substitute $2x - 20x$ for bx to get $5x^2 + 2x - 20x - 8$.
Factor by grouping $(5x^2 + 2x) - (20x + 8) = x(5x + 2) - 4(5x + 2) = (5x + 2)(x - 4)$.
Check: $(5x + 2)(x - 4) = 5x^2 - 20x + 2x - 8 = 5x^2 - 18x - 8$

DIRECTIONS: Factor the trinomial. If the trinomial cannot be factored, write "cannot be factored."

1. $2x^2 + 7x + 6$

2. $3x^2 + 14x - 5$

3. $3x^2 + 7x + 1$

4. $8x^2 + 2x - 3$

5. $6x^2 + 7x - 3$

CHALLENGE: When factoring $6x^2 - x - 1$, Ronnie said that he found all the factors of 6. He said that the polynomial could not be factored because there is no pair of factors of 6 that has a sum equal to -1. Is he correct? Explain your reasoning.

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Teaching Notes 5.19: Factoring the Sums and Differences of Cubes

Factoring the sums and differences of cubes often presents difficulties for students. Some students have trouble identifying the cubes of numbers and others confuse the formulas.

1. Explain that the sums and differences of cubes are binomials and that there are specific formulas for factoring the sums of cubes and the differences of cubes.
2. Explain that students must be familiar with cubes of numbers in order to apply these formulas. Remind them that the cube of a number is a number raised to the third power. Offer these examples: $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, $5^3 = 125$, $6^3 = 216$, $7^3 = 343$, $8^3 = 512$, $9^3 = 729$, and $10^3 = 1,000$. You might find it helpful to expand the notation, for example, $4^3 = 4 \cdot 4 \cdot 4 = 64$.
3. Review the formulas, information, and example on the worksheet with your students. Especially emphasize that they can check their work by multiplying the factors. Depending on the abilities of your students, you might find 5.9: "Multiplying Two Polynomials" helpful.

EXTRA HELP:

The formulas on the worksheet only apply to the sums or differences of cubes.

ANSWER KEY:

(1) $(x - 2)(x^2 + 2x + 4)$ **(2)** $(x + 5)(x^2 - 5x + 25)$ **(3)** Cannot be factored
(4) $(x - 1)(x^2 + x + 1)$ **(5)** $(4 - y)(16 + 4y + y^2)$ **(6)** $(xy + 8)(x^2y^2 - 8xy + 64)$

(Challenge) Jasmine's factors were correct. The check was wrong. She should not have used FOIL to check her work. Therefore, her answer did not check.

Name _____ Date _____

WORKSHEET 5.19: FACTORING THE SUMS AND DIFFERENCES OF CUBES

Binomials that are the sums and differences of cubes can be factored by using these formulas:

Sum of cubes	Difference of cubes
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

To use these formulas, follow the steps below:

1. Identify the cubed terms in each binomial.
2. Substitute the appropriate variable for a and b in the correct formula.
3. Check your work by multiplying the factors.

EXAMPLE

Factor $x^3 - 27$.

x is cubed and 27 is equal to 3^3. The expression is the difference of cubes.

Use the formula for finding the difference of cubes. Substitute x for a and 3 for b in the formula. $x^3 - 3^3 = (x - 3)(x^2 + 3x + 9)$

Check: $x(x^2 + 3x + 9) - 3(x^2 + 3x + 9) = x^3 + 3x^2 + 9x - 3x^2 - 9x - 27 = x^3 - 27$

DIRECTIONS: Factor each binomial. If the binomial cannot be factored, write "cannot be factored."

1. $x^3 - 8$
2. $x^3 + 125$
3. $x^3 + 32$
4. $x^3 - 1$
5. $64 - y^3$
6. $x^3y^3 + 512$

CHALLENGE: Jasmine factored $x^3 + 27$ as $(x + 3)(x^2 - 3x + 9)$. She used FOIL to check the factors and found $x^3 + 9x + 3x^2 + 27$. Her work did not check. What was her mistake?

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Teaching Notes 5.20: Solving Quadratic Equations by Factoring

Factoring is one way to solve a quadratic equation. To use this method, students must factor the equation and use the zero-product property. Many students forget to set the equation equal to zero and consequently find an incorrect solution to the equation.

1. Review factoring polynomials with your students by providing these prompts:
   - Is there a greatest monomial factor?
   - Is the polynomial the difference of squares?
   - Is the polynomial factorable?
2. Review the information and example on the worksheet with your students. Be sure to explain the zero-product property and show students how to apply this property to find the solutions of the equation. Emphasize that each factor must be written as an equation that is equal to zero. Students can then solve for the variable. If necessary, discuss in detail the steps for solving the example equation by factoring.

EXTRA HELP:

Have students each write an equation that states that each factor is equal to zero. This will help them solve the equations accurately.

ANSWER KEY:

(1) $x = -4$ and $x = -2$ **(2)** $x = 3$ and $x = 1$ **(3)** $x = -5$ and $x = 5$ **(4)** $x = 4$ and $x = 6$

(5) $x = -1\frac{1}{2}$ and $x = -1$ **(6)** $x = 1$ and $x = 1\frac{1}{3}$ **(7)** $x = -7$ and $x = -2$ **(8)** $x = -2$ and $x = 3$

(Challenge) Kim needs to subtract 56 and $7x$ from both sides of the equation. This will create the new equation of $x^2 + x - 56 = 0$. She can then factor the equation and solve for x. The solutions are $x = 7$ and $x = -8$.

Name _____ Date _____

WORKSHEET 5.20: SOLVING QUADRATIC EQUATIONS BY FACTORING

To solve a quadratic equation by factoring, you must first factor the equation and then use the zero-product property. The zero-product property states that if a and b are real numbers and $ab = 0$, then $a = 0$ or $b = 0$. Follow the steps below:

1. Write the equation so that it is equal to zero (if this is not already done).
2. Factor one side of the equation.
3. Set each factor equal to zero.
4. Solve each equation.
5. Substitute your solutions into the original equation to check your work.

EXAMPLE

Solve $x^2 + 2x = 15$ by factoring.

$$x^2 + 2x - 15 = 0$$
$$(x - 3)(x + 5) = 0$$
$$(x - 3) = 0 \text{ or } (x + 5) = 0$$
$$x = 3 \quad \text{or} \quad x = -5$$
$$(3)^2 + 2(3) = 15 \text{ and } (-5)^2 + 2(-5) = 15$$

DIRECTIONS: Solve each equation by factoring.

1. $x^2 + 6x + 8 = 0$
2. $2x^2 - 8x + 6 = 0$
3. $x^2 - 25 = 0$
4. $x^2 - 10x + 24 = 0$
5. $2x^2 + 5x + 3 = 0$
6. $6x^2 - 14x = -8$
7. $x^2 + 9x = -14$
8. $4x^2 - 4x = 24$

CHALLENGE: Kim was asked to solve $x^2 + 8x = 7x + 56$. She did not know how to solve it because the equation is not equal to zero. What should she do to solve this problem?

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Teaching Notes 5.21: Solving Quadratic Equations by Finding Square Roots

Solving quadratic equations by finding square roots involves a basic understanding of square roots. Students may forget that positive numbers have two square roots, zero has one square root, and negative numbers have no real square roots.

1. Discuss the concept of squares and square roots with your students. Ask them what number multiplied by itself equals 9. Most students will say 3×3, which is correct. Be sure to note that -3×-3 also equals 9. Both examples may be expressed as $x^2 = 9$.
2. Explain that because 9 is a positive number, $x^2 = 9$ has two solutions. The two solutions are $x = \pm\sqrt{9}$, which means $x = 3$ and $x = -3$. Emphasize that the symbol $\pm$ is read "plus or minus" and denotes both the positive and negative square roots.
3. Present additional examples. Ask your students what number multiplied by itself equals 0. There is only one solution, $x = 0$. Ask them what number multiplied by itself equals -49. There is no real solution, because no real number multiplied by itself is equal to -49. (You might want to mention that there are solutions for negative numbers in the imaginary numbers.)
4. Review the information and example on the worksheet with your students. Note that students must isolate the squared term on one side of the equation before they can find the square root.

EXTRA HELP:

This method for solving quadratic equations by finding square roots cannot be used if there is an x term in the equation.

ANSWER KEY:

(1) $x = 11$ and $x = -11$ **(2)** $x = 10$ and $x = -10$ **(3)** $x = 1$ and $x = -1$ **(4)** $x = 0$
(5) No real solutions **(6)** $x = 7$ and $x = -7$ **(7)** $x = 0$ **(8)** No real solutions

(Challenge) Deanna rewrote the problem incorrectly. She should have subtracted 16 from both sides of the equation and written $x^2 = -16$. When she substituted her answers in an incorrect equation they checked. The answers would not have checked if she had substituted them in the original equation.

Name _____ Date _____

WORKSHEET 5.21: SOLVING QUADRATIC EQUATIONS BY FINDING SQUARE ROOTS

Follow the steps below to solve a quadratic equation of the form $ax^2 + c = 0$, where a cannot equal 0:

1. Isolate the squared term on one side of the equation.
2. Find the square root of the terms on both sides of the equation. Remember the following:
   - All positive numbers have two square roots. One is positive; the other is negative.
   - The square root of 0 is 0.
   - All negative numbers have no real square roots.
3. Substitute your solution, if there is one, in the original equation to check your work.

EXAMPLE

Solve $2x^2 - 50 = 0$.
To isolate x^2, add 50 to both sides of the equation, then divide by 2. The equivalent equation is $x^2 = 25$.
Find the square root of each side. $\sqrt{x^2} = \pm\sqrt{25}$. $x = 5$; $x = -5$
Check: $2(5)^2 - 50 = 0$ and $2(-5)^2 - 50 = 0$

DIRECTIONS: Solve. If there are no real solutions, write "no real solutions."

1. $x^2 = 121$
2. $4x^2 = 400$
3. $x^2 - 1 = 0$
4. $3x^2 = 0$
5. $x^2 + 49 = 0$
6. $-4x^2 + 196 = 0$
7. $x^2 + 49 = 49$
8. $-5x^2 - 20 = 0$

CHALLENGE: Deanna solved $x^2 + 16 = 0$ by writing $x^2 = 16$. She then found the square root of both sides of the equation and found that $x = 4$ and $x = -4$. She substituted both of these values in $x^2 = 16$ and both answers checked. But her teacher told her that there are no real solutions to the original equation. What was Deanna's mistake?

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Teaching Notes 5.22: Solving Quadratic Equations Using the Quadratic Formula

The quadratic formula can be used to solve quadratic equations. Students often have trouble using this formula because of the three variables, square roots, and fractions.

1. Explain that the quadratic formula can be used to solve quadratic equations. But before it can be used the equation must be written in standard form. $ax^2 + bx + c = 0$ where $a \neq 0$.
2. Review the information and example on the worksheet with your students. Be sure that they understand both the quadratic formula and the standard form of a quadratic equation. Note that the equation in the example is already written in standard form. (If necessary, review how to express an equation in standard form by adding the same term to or subtracting the same term from both sides of the equation.) Point out that the expression inside the radical symbol must be greater than or equal to 0 because the square root of a negative number is not a real number. Discuss the steps for using the quadratic formula to solve an equation. Depending on the abilities of your students, you may find it necessary to demonstrate the order of operations.

EXTRA HELP:

Use the order of operations to solve for x. Simplify the expression inside the radical symbol, find the square root, simplify the numerator, and then divide.

ANSWER KEY:

(1) $x = 2$ and $x = 3$ **(2)** $x = 4$ and $x = -1$ **(3)** $x = -3$ and $x - 6$ **(4)** $x = -5$

(Challenge) You would first need to rewrite the equation in the standard form by adding x to both sides of the equation and subtracting 12 from both sides of the equation. Then you could solve the equation using the quadratic formula, substituting 1 for a, 1 for b, and -12 for c. The correct solutions are $x = -4$ and $x = 3$.

Name _____ Date _____

WORKSHEET 5.22: SOLVING QUADRATIC EQUATIONS USING THE QUADRATIC FORMULA

The quadratic formula states that $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $a \neq 0$ and $b^2 - 4ac \geq 0$. You can use the quadratic formula to solve a quadratic equation that is written in standard form, $ax^2 + bx + c = 0$, where $a \neq 0$. Follow the steps below:

1. Write the equation in standard form. Identify the numbers that correspond to the variables a, b, and c. Substitute these values in the quadratic formula.
2. Write the quadratic formula as two equations: one that adds the square root and one that subtracts the square root.
3. Simplify using the order of operations to find the solutions.
4. Substitute these solutions in the original equation to check your work.

EXAMPLE

$x^2 + 3x + 2 = 0$ Use the quadratic formula to solve for x. $a = 1, b = 3, c = 2$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} \rightarrow x = \frac{-3 + \sqrt{3^2 - 4(1)(2)}}{2(1)} \text{ or } x = \frac{-3 - \sqrt{3^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-3 + \sqrt{9 - 8}}{2} \text{ or } x = \frac{-3 - \sqrt{9 - 8}}{2}$$

$x = -1$ and $x = -2$ Check: $(-1)^2 + 3(-1) + 2 = 0$ and $(-2)^2 + 3(-2) + 2 = 0$

DIRECTIONS: Use the quadratic formula to solve each equation.

1. $x^2 - 5x + 6 = 0$

2. $x^2 - 3x - 4 = 0$

3. $x^2 + 9x + 18 = 0$

4. $x^2 + 10x = -25$

CHALLENGE: Explain how you would solve $x^2 = -x + 12$ by using the quadratic formula.

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Teaching Notes 5.23: Using the Discriminant

The discriminant provides students with information about the number of solutions of a quadratic equation. Students often make computational mistakes when finding the discriminant or they confuse the number of solutions if the discriminant is positive, negative, or zero.

1. Review the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, with your students.
2. Review the information and examples on the worksheet with your students. Emphasize that the expression inside the radical symbol, $b^2 - 4ac$, is called the discriminant and can be evaluated by using the order of operations. Its value indicates the number of solutions to a quadratic equation. Note that the examples are written in standard form. Also explain the following:
   - There is one solution when the discriminant is 0. When 0 is substituted for the discriminant in the quadratic formula, $\dfrac{-b}{2a}$ has only one value.
   - There are two solutions if the discriminant is positive: the quantity $-b$ plus the square root of the discriminant divided by $2a$ and the quantity $-b$ minus the square root of the discriminate divided by $2a$.
   - There are no real solutions if the discriminant is negative. The square root of a negative number is not a real number.

EXTRA HELP:

Be sure that the equation is in standard form before you find the value of the discriminant.

ANSWER KEY:

(1) Two solutions **(2)** No real solutions **(3)** Two solutions **(4)** One solution

(Challenge) It is important to find the discriminant before solving a quadratic equation because the discriminant allows you to know how many solutions you are looking for.

Name _____ Date _____

WORKSHEET 5.23: USING THE DISCRIMINANT

The discriminant, $b^2 - 4ac$, is the expression that is inside the radical symbol in the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The discriminant can be used to determine the number of solutions of a quadratic equation expressed in standard form as $ax^2 + bx + c = 0$, where $a \neq 0$. Use the guidelines below:

1. Write the equation in standard form, if necessary.
2. Identify the values of a, b, and c.
3. Substitute these values in the discriminant and use the order of operations to find the value of the discriminant.
   - If the value of the discriminant is 0, the equation has one solution.
   - If the value of the discriminant is positive, the equation has two solutions.
   - If the value of the discriminant is negative, the equation has no real solutions.

Example 1	Example 2	Example 3
$x^2 - 4x + 4 = 0$	$x^2 + 5x + 3 = 0$	$2x^2 + x + 5 = 0$
$a = 1, b = -4, c = 4$	$a = 1, b = 5, c = 3$	$a = 2, b = 1, c = 5$
$(-4)^2 - 4(1)(4)$	$5^2 - 4(1)(3)$	$1^2 - 4(2)(5)$
$16 - 6$	$25 - 12$	$1 - 40$
0	13	-39
There is one solution.	There are two solutions.	There are no real solutions.

DIRECTIONS: Use the discriminant to determine the number of solutions. If there are no real solutions, write "no real solutions."

1. $x^2 + 6x - 5 = 0$

2. $x^2 - 4x + 10 = 0$

2. $x^2 - 2x - 3 = 0$

4. $x^2 + 6x = -9$

CHALLENGE: Why is finding the discriminant an important step to take before solving an equation?

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SECTION 6

Rational Expressions

Teaching Notes 6.1: Using Zero and Negative Numbers as Exponents

Students often find using zero and negative numbers as exponents confusing. Negative exponents are particularly troublesome because students seldom connect negative numbers with reciprocals.

1. Depending on the abilities of your students, you may want to begin with a review of exponents. Explain that an exponent is the number of times the base is used as a factor. You might want to provide an example, such as $2^3 = 2 \times 2 \times 2 = 8$.
2. To help your students discover how zero and negative numbers are used as exponents, present the example of 2^n. Have your students evaluate the expression when $n = 3$, 2, and 1. The answers are 8, 4, and 2. Note that each number is half of the previous term. Therefore, the next term is $2^0 = 1$. Using the same reasoning, $2^{-1} = \frac{1}{2}$ and $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$, and so on.
3. Review the table, definitions, and examples on the worksheet with your students. Discuss the definitions of zero and negative exponents. Especially note the meaning of the negative exponent in the base of the denominator in the last example.

EXTRA HELP:

Pay close attention to the base. If the base is a negative number, the base must be written in parentheses.

ANSWER KEY:

(1) $\frac{1}{5^2} = \frac{1}{25}$ **(2)** 1 **(3)** $\frac{1}{(-3)^4} = \frac{1}{81}$ **(4)** $-\frac{1}{3^4} = -\frac{1}{81}$ **(5)** 1 **(6)** $7^2 = 49$

(7) $-\frac{1}{4^2} = -\frac{1}{16}$ **(8)** $\frac{1}{(-4)^2} = \frac{1}{16}$ **(9)** $\frac{8}{3}$ **(10)** $6^2 = 36$ **(11)** $\frac{1}{(-2)^3} = \frac{1}{-8}$ **(12)** $\frac{1}{2^3} = \frac{1}{8}$

(Challenge) They are different expressions and have different values. $-5^0 = -1$ and $(-5)^0 = 1$

Name _____ Date _____

WORKSHEET 6.1: USING ZERO AND NEGATIVE NUMBERS AS EXPONENTS

Examine the table below to discover patterns involving 3^n. The numbers in the top row represent exponents of 3. Because the base is 3, each term is $\frac{1}{3}$ of the previous term.

n	3	2	1	0	-1	-2	-3
3^n	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

Following is the definition of zero as an exponent. If x is a nonzero number, then $x^0 = 1$.

EXAMPLES

$$4^0 = 1 \qquad (-2)^0 = 1 \qquad \left(\frac{1}{5}\right)^0 = 1$$

Following are the definitions of negative exponents. If x is a nonzero number, then $x^{-n} = \frac{1}{x^n}$ and $\frac{1}{x^{-n}} = x^n$

EXAMPLES

$$4^{-2} = \frac{1}{4^2} = \frac{1}{16} \qquad \frac{1}{5^{-2}} = 5^2 = 25$$

DIRECTIONS: Simplify each expression.

1. 5^{-2} **2.** 6^0 **3.** $(-3)^{-4}$ **4.** -3^{-4}

5. $\left(\frac{2}{3}\right)^0$ **6.** $\frac{1}{7^{-2}}$ **7.** -4^{-2} **8.** $(-4)^{-2}$

9. $\left(\frac{3}{8}\right)^{-1}$ **10.** $\frac{1}{6^{-2}}$ **11.** $(-2)^{-3}$ **12.** 2^{-3}

CHALLENGE: Is -5^0 the same as $(-5)^0$. Explain your answer.

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Teaching Notes 6.2: Using the Properties of Exponents That Apply to Division

To accurately simplify expressions, students may need to use the properties of exponents for division. Students often forget these properties and simplify expressions incorrectly.

1. Discuss the basic steps for simplifying expressions with exponents. For example, explain that $2^3 = 2 \cdot 2 \cdot 2 = 8$. Emphasize that 2 is the base and 3 is the exponent. The exponent 3 means that the base is multiplied by itself three times.
2. Explain that there are two properties for working with exponents and division: the property of exponents for division and the property of exponents for a power of a quotient.
3. Review the properties and examples on the worksheet with your students. Be sure that they understand how and when to apply these properties. Note that in the property of exponents for division, the base is the same and students must subtract the exponent of the base in the denominator from the exponent of the base in the numerator. Note that in the property of exponents for a power of a quotient, the bases are different and the exponent applies to the bases in both the numerator and denominator. Caution your students to be particularly careful with negative exponents.

EXTRA HELP:

When dividing powers that have the same base, always subtract the exponent of the denominator from the exponent of the numerator.

ANSWER KEY:

(1) $\frac{3^3}{4^3} = \frac{27}{64}$ **(2)** $\frac{1^4}{4^4} = \frac{1}{256}$ **(3)** $2^1 = 2$ **(4)** $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

(5) $\frac{5^{-2}}{6^{-2}} = \frac{6^2}{5^2} = \frac{36}{25}$ **(6)** $5^{-1} = \frac{1}{5^1} = \frac{1}{5}$ **(7)** $8^{-1} = \frac{1}{8^1} = \frac{1}{8}$ **(8)** $3^{-6} = \frac{1}{3^6} = \frac{1}{729}$

(Challenge) Francine and Phil are both correct. They used different procedures to get the same answer.

Name _____ Date _____

WORKSHEET 6.2: USING THE PROPERTIES OF EXPONENTS THAT APPLY TO DIVISION

There are two properties for working with exponents and division.

Property of exponents for division: To divide powers having the same base, subtract the exponent in the denominator from the exponent in the numerator.

$$\frac{x^m}{x^n} = x^{m-n} \qquad x \neq 0$$

EXAMPLES

$$\frac{4^7}{4^2} = 4^{7-2} = 4^5 = 1,024 \qquad \frac{5^2}{5^5} = 5^{2-5} = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$$

Property of exponents for a power of a quotient: To find the power of a quotient, find the power of the numerator and the power of the denominator.

$$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m} \qquad y \neq 0$$

EXAMPLES

$$\left(\frac{5}{7}\right)^3 = \frac{5^3}{7^3} = \frac{125}{343} \qquad \left(\frac{2}{5}\right)^{-4} = \frac{2^{-4}}{5^{-4}} = \frac{5^4}{2^4} = \frac{625}{16}$$

DIRECTIONS: Rewrite each expression using the properties of exponents for division. Then simplify each expression.

1. $\left(\frac{3}{4}\right)^3$ **2.** $\left(\frac{1}{4}\right)^4$ **3.** $\frac{2^3}{2^2}$ **4.** $\frac{3^4}{3^6}$

5. $\left(\frac{5}{6}\right)^{-2}$ **6.** $\frac{5^2}{5^3}$ **7.** $\frac{8^5}{8^6}$ **8.** $\frac{3^{-2}}{3^4}$

CHALLENGE: Francine applied the property of exponents for division to simplify $\frac{8^5}{8^5}$. She said that $8^{5-5} = 8^0 = 1$. Phil said that $\frac{8^5}{8^5}$ is a fraction whose numerator and denominator are the same and therefore the fraction is equal to 1. Who is correct? Explain your reasoning.

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Teaching Notes 6.3: Using the Properties of Exponents That Apply to Multiplication and Division

There are five different properties for working with exponents. Confusion arises when students are asked to apply several properties to evaluate one expression. A review of the properties is helpful.

1. Explain that several properties for exponents may be applied to evaluate an expression. It is important to know when to apply each one.
2. Review the properties and examples on the worksheet with your students. Share the following tips:
   - *Multiplication:* Add the exponents if the bases are the same.
   - *Power of a power:* Multiply the exponents.
   - *Power of a product:* Raise each base of the product to the power.
   - *Division:* Subtract the exponent of the base in the denominator from the exponent of the base in the numerator, provided the bases are the same.
   - *Power of a quotient:* Raise each base of the quotient to the power.

EXTRA HELP:

If there are several operations in the same problem, follow the order of operations.

ANSWER KEY:

(1) $2^6 = 64$ **(2)** $2^{-2} = \frac{1}{4}$ **(3)** $\frac{5^3}{5^3} = 1$ **(4)** $6^{-2} = \frac{1}{36}$

(5) $5^3 \cdot 2^3 = 125 \cdot 8 = 1,000$ **(6)** $7^5 = 16,807$ **(7)** $\frac{2^3}{5^3} = \frac{8}{125}$ **(8)** $\frac{3^6 \cdot 2^3}{5^3} = \frac{729 \cdot 8}{125} = \frac{5,832}{125}$

(Challenge) Both are correct. They used a correct property and followed the correct procedures. If both expressions were expressed in simplest form, both would equal 576.

Name _____ Date _____

WORKSHEET 6.3: USING THE PROPERTIES OF EXPONENTS THAT APPLY TO MULTIPLICATION AND DIVISION

The properties of exponents are summarized below:

- Multiplication: $x^m x^n = x^{m+n}$ *Example:* $2^5 \cdot 2^{-8} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
- Power of a power: $(x^m)^n = x^{mn}$ *Example:* $(3^2)^3 = 3^{2 \cdot 3} = 3^6 = 729$
- Power of a product: $(xy)^m = x^m y^m$ *Example:* $(2 \cdot 3)^2 = 2^2 \cdot 3^2 = 4 \cdot 9 = 36$
- Division: $\frac{x^m}{x^n} = x^{m-n}; x \neq 0$ *Example:* $\frac{2^3}{2^{-1}} = 2^{3-(-1)} = 2^4 = 16$
- Power of a quotient: $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}; y \neq 0$ *Example:* $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$

DIRECTIONS: Simplify.

1. $2^2 \cdot 2^4$

2. $2^{-3} \cdot 2^1$

3. $\frac{5^2 \cdot 5^1}{5^3}$

4. $(6^{-1})^2$

5. $(5 \cdot 2)^3$

6. $\frac{7^2}{7^{-3}}$

7. $\left(\frac{2}{5}\right)^3$

8. $\left(\frac{3^2 \cdot 2}{5}\right)^3$

CHALLENGE: Mike used the power of a product property to simplify $(2^3 \cdot 3)^2$ as $2^6 \cdot 3^2$. Nicole used the order of operations and simplified $(2^3 \cdot 3)^2$ as $(8 \cdot 3)^2$ or 24^2. Who was correct? Explain your answer.

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Teaching Notes 6.4: Identifying Restrictions on the Variable

The value of the denominator of any fraction cannot equal 0. When students work with algebraic fractions, they must identify those values that would make the denominator equal to 0 and exclude them from the solution. This concept, placing a restriction on a variable, can be confusing, especially because students must factor the denominator and identify the values that would make the denominator equal to 0.

1. Explain to your students that a fraction such as $\frac{5}{0}$ is undefined. To show your students why this is so, provide examples of division problems such as $\frac{-12}{3} = -4$. Note that students can check this answer by multiplying $-4 \cdot 3 = -12$. Now provide an example such as $\frac{5}{0}$ and ask students to provide an answer that they think is correct. One common response is 5. Note that $0 \cdot 5 \neq 5$. The answer does not check, nor will any other. Using this reasoning, students will realize that division by 0 is impossible.
2. Review the information and examples on the worksheet with your students. Emphasize that if there is a variable in the denominator, students must restrict the variables in the denominator so that the denominator does not equal zero. Discuss the procedure for identifying restrictions on variables in detail. Note that if the highest power of the variable is 1, students should write and solve an inequality that states that the denominator does not equal 0. If the highest power of the variable is 2 or more, students must factor the denominator. If at least one of the factors is equal to 0, the denominator is equal to 0. Discuss the examples, making certain that your students understand how to solve the inequalities.

EXTRA HELP:

The numerator has no effect on the restrictions of the variable.

ANSWER KEY:

(1) $x \neq 4$ **(2)** $x \neq 0$ **(3)** $x \neq 1; x \neq -1$ **(4)** $x \neq -4; x \neq 1$

(5) $x \neq -1; x \neq 11$ **(6)** $x \neq -7; x \neq 1$ **(7)** $x \neq 0; x \neq 1$ **(8)** $x \neq -8; x \neq -5$

(Challenge) The fraction equals 1, provided that $x \neq 2$. When $x = 2$, the denominator is equal to 0 and the fraction is undefined.

Name _____ Date _____

WORKSHEET 6.4: IDENTIFYING RESTRICTIONS ON THE VARIABLE

A restriction on a variable in the denominator of a fraction is a value that the variable may not equal. Otherwise the fraction will have no meaning. Follow the steps below to identify restrictions on a variable:

1. If the highest degree of the variable in the denominator is 1, write an inequality, stating that the denominator cannot equal 0.
2. If the degree of the variable in the denominator is 2 or more, factor the denominator and write an inequality, stating that each factor cannot equal 0.
3. Solve each inequality to find the restrictions on the variable.

EXAMPLES

Find restrictions on the variables below.

$\frac{3}{x+1}$ Because the exponent of x is 1, write and solve the inequality: $x + 1 \neq 0; x \neq -1$.

$\frac{4}{x^2 - 6x + 5}$ Factor $x^2 - 6x + 5$: $x^2 - 6x + 5 = (x - 5)(x - 1)$. Write the inequalities: $x - 5 \neq 0$ and $x - 1 \neq 0$. Solve each inequality: $x \neq 5$ and $x \neq 1$.

DIRECTIONS: Find the restrictions on the variable.

1. $\frac{3}{x-4}$

2. $\frac{2x}{4x}$

3. $\frac{x-1}{x^2-1}$

4. $\frac{5x^2}{x^2+3x-4}$

5. $\frac{x^2-4}{x^2-10x-11}$

6. $\frac{1}{x^2+6x-7}$

7. $\frac{8}{x^2-x}$

8. $\frac{x^2+3x}{x^2+13x+40}$

CHALLENGE: Marty said $\frac{x-2}{x-2} = 1$. Do you agree? Explain your answer.

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Teaching Notes 6.5: Simplifying Algebraic Fractions

Some students incorrectly apply the cancellation rule when simplifying an algebraic fraction. Fully understanding the process eliminates this common error.

1. Explain that some algebraic fractions, fractions that have a variable in the numerator and/or denominator, may be simplified using the same process that is used for simplifying fractions.
2. Present this example to your students: $\frac{10}{15} = \frac{2 \cdot \cancel{5}}{3 \cdot \cancel{5}} = \frac{2}{3}$ Note that the common factor, 5, may be cancelled because $\frac{5}{5} = 1$. Explain that this same process applies to simplifying the quotients of polynomials by factoring and cancelling the common factors, provided that the denominator does not equal 0. If the denominator equals 0, the fraction is undefined.
3. Review the information and examples on the worksheet with your students. Explain that only the factors of a polynomial that have the same terms may be cancelled. Note the first example: $\frac{8 + 4x}{6x + 12} = \frac{4(2 + x)}{6(x + 2)}$. In this example, $\frac{2 + x}{x + 2}$ can be cancelled because addition is commutative and this fraction is equal to 1. Emphasize that in the second example students cannot cancel the x^2 term because x^2 is not a factor of the polynomial. Students must factor the trinomial.

EXTRA HELP:

The quotient of polynomials is in simplest form if the only common factor is 1 or -1.

ANSWER KEY:

(1) $\frac{x - 4}{6x - 2}$ **(2)** $\frac{1}{2}$ **(3)** $\frac{5}{6}$ **(4)** $\frac{2x + 1}{6x + 4}$

(5) $x + y$ **(6)** $\frac{x + 3}{x + 4}$ **(7)** $\frac{3x - 4}{1 + x}$ **(8)** $x - 5$

(Challenge) No. It can be simplified further as $\frac{2x - 8}{x + 5}$.

Name _____ Date _____

WORKSHEET 6.5: SIMPLIFYING ALGEBRAIC FRACTIONS

You can simplify algebraic fractions by factoring and cancelling the common factors. Assume that the denominator of the fraction is not equal to 0.

EXAMPLES

$$\frac{8+4x}{6x+12} = \frac{4(2+x)}{6(x+2)} = \frac{\cancel{2}\cdot 2(\cancel{2+x})}{\cancel{2}\cdot 3(\cancel{x+2})} = \frac{2}{3}$$

$$\frac{x^2+11x+24}{x^2+5x+6} = \frac{(x+8)\,(\cancel{x+3})}{(x+2)\,(\cancel{x+3})} = \frac{(x+8)}{(x+2)}$$

DIRECTIONS: Simplify each fraction.

1. $\dfrac{2x-8}{12x-4}$

2. $\dfrac{4x+1}{8x+2}$

3. $\dfrac{5x-5y}{6x-6y}$

4. $\dfrac{2x^3+x^2}{6x^3+4x^2}$

5. $\dfrac{x^2-y^2}{x-y}$

6. $\dfrac{x^2+x-6}{x^2+2x-8}$

7. $\dfrac{3x^2+5x-12}{3+4x+x^2}$

8. $\dfrac{x^2-10x+25}{x-5}$

CHALLENGE: Do you agree that $\dfrac{2x^2-10x+8}{x^2+4x-5}$ is expressed in simplest form as $\dfrac{2(x^2-5x+4)}{x^2+4x-5}$? Explain your answer.

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Teaching Notes 6.6: Adding and Subtracting Algebraic Fractions with Like Denominators

Adding and subtracting algebraic fractions often results in either of two problems. Students may add or subtract the terms in the numerator incorrectly or they may apply the cancellation rule incorrectly.

1. Demonstrate addition of fractions with the following example: $\frac{3}{20}+\frac{1}{20}=\frac{4}{20}=\frac{4}{5\times 4}=\frac{1}{5}$. Note that students factored 20 either mentally or on paper to simplify the fraction.
2. Demonstrate subtraction of fractions with the following example: $\frac{3}{8}-\left(-\frac{1}{8}\right)=\frac{3}{8}+\frac{1}{8}=\frac{4}{8}=\frac{4}{2\times 4}=\frac{1}{2}$. Note that students must rewrite subtraction as addition of the opposite term and then proceed to add and simplify.
3. Emphasize that the process of adding and subtracting algebraic fractions is similar to the examples, except that students will add or subtract numbers and variables. They may also have to factor polynomials to simplify the fraction. You may wish to review 6.5: "Simplifying Algebraic Fractions."
4. Review the information and examples on the worksheet with your students. Depending on their abilities, you may find it necessary to go through the steps of the examples in detail. Be sure that your students understand how to simplify using cancellation.

EXTRA HELP:

Rewrite subtraction by adding the opposite of every term in the numerator of the second fraction.

ANSWER KEY:

(1) $x-1$ **(2)** $\frac{1}{2}$ **(3)** $\frac{1}{x+4}$ **(4)** $\frac{x+4}{x-2}$ **(5)** $\frac{x+3}{x+5}$ **(6)** $\frac{x-5}{x-1}$

(Challenge) Julian's error is in subtraction. The correct answer is $\frac{x^2-7x-6}{x^2+x-2}$, which cannot be simplified.

Name _____ Date _____

WORKSHEET 6.6: ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS WITH LIKE DENOMINATORS

Follow the steps below to add and subtract algebraic fractions with like denominators. Assume that the denominators of the fractions are not equal to zero.

1. Add or subtract the numerators. If you are subtracting, add the opposite of each term in the numerator of the second fraction.
2. Place the sum or difference over the denominator.
3. Factor the numerator and the denominator, if possible.
4. Use the cancellation rule to simplify.

EXAMPLES

$$\frac{x}{x^2+3x+2}+\frac{2}{x^2+3x+2}=\frac{x+2}{x^2+3x+2}=\frac{x+2}{(x+2)(x+1)}=\frac{1}{x+1}$$

$$\frac{x^2}{x^2-4x+5}-\frac{2x+3}{x^2-4x+5}=\frac{x^2-2x-3}{x^2-4x-5}=\frac{(x-3)(x+1)}{(x-5)(x+1)}=\frac{x-3}{x-5}$$

DIRECTIONS: Find the sum or difference. Simplify.

1. $\dfrac{x^2}{x+1}-\dfrac{1}{x+1}$

2. $\dfrac{3x}{6x+8}+\dfrac{4}{6x+8}$

3. $\dfrac{x}{x^2-x-20}-\dfrac{5}{x^2-x-20}$

4. $\dfrac{x^2}{x^2-3x+2}+\dfrac{3x-4}{x^2-3x+2}$

5. $\dfrac{x^2+8x}{x^2+10x+25}+\dfrac{15}{x^2+10x+25}$

6. $\dfrac{x^2}{x^2-2x+1}-\dfrac{6x-5}{x^2-2x+1}$

CHALLENGE: Julian said that $\dfrac{x^2}{x^2+x-2}-\dfrac{7x+6}{x^2+x-2}=\dfrac{x^2-7x+6}{x^2+x-2}$. He simplified his answer as $\dfrac{x-6}{x+2}$. Is he correct? Explain your answer.

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Teaching Notes 6.7: Finding the Least Common Multiple of Polynomials

The concept of finding the least common multiple of polynomials is essential for finding the least common denominator of algebraic fractions. Errors occur when students factor a polynomial incorrectly or when they do not consider all of the common factors.

1. Explain the process for finding the least common multiple (LCM) of whole numbers to your students. Present this example: Find the LCM of 20 and 75. Note that students are to find the prime factorization of each number and express the prime factorization using exponents. $20 = 2^2 \times 5$; $75 = 5^2 \times 3$ Remind your students that the LCM is expressed as the product of each base raised to the highest power of each factor. Therefore, the LCM of 20 and 75 is $2^2 \times 5^2 \times 3$ or 300. You might find it helpful to review 1.10: "Writing Prime Factorization."
2. Explain that the process of finding the LCM of algebraic fractions follows the same procedure. But instead of factoring numbers, students will be factoring polynomials.
3. Review the information and example on the worksheet with your students. Be sure that your students understand each step of the example.

EXTRA HELP:

Use all of the factors, not just the ones that are common to both polynomials when writing the LCM.

ANSWER KEY:

(1) $(x-2)(x-3)(x-4)$ **(2)** $(x-2)(x+4)(x-4)$ **(3)** $(x-5)^2(x+4)$ **(4)** $(x-1)(x+4)(x+2)$

(5) $(x+4)^2(x+1)$ **(6)** $x^2(x+1)(x+6)$ **(7)** $5(x^2+10)$ **(8)** $(x-1)(x+4)(x+8)(x+2)$

(Challenge) Deena is correct. If two polynomials (or whole numbers) have no common factors other than 1, the LCM is the product of the factors.

Name _____ Date _____

WORKSHEET 6.7: FINDING THE LEAST COMMON MULTIPLE OF POLYNOMIALS

Follow the steps below to find the least common multiple (LCM) of polynomials:

1. Factor each polynomial.
2. Write the product of each factor, raised to the highest number of times it is a factor of one of the polynomials.

EXAMPLE

Find the LCM of $x^2 - 4$ and $x^2 - 4x + 4$.

Factor. $x^2 - 4 = (x - 2)(x + 2)$ and $x^2 - 4x + 4 = (x - 2)(x - 2)$ or $(x - 2)^2$.

$(x - 2)$ and $(x + 2)$ are the factors. The highest power of $(x + 2)$ is 1. The highest power of $(x - 2)$ is 2. Therefore, $(x + 2)(x - 2)^2$ is the LCM.

DIRECTIONS: Find the LCM of the polynomials.

1. $x^2 - 5x + 6$ and $x^2 - 6x + 8$

2. $x^2 - 16$ and $x^2 + 2x - 8$

3. $x^2 - 10x + 25$ and $x^2 - x - 20$

4. $x^2 + 3x - 4$ and $x^2 + 6x + 8$

5. $x^2 + 8x + 16$ and $x^2 + 5x + 4$

6. $x^3 + x^2$ and $x^2 + 7x + 6$

7. $5x^2 + 50$ and $x^2 + 10$

8. $x^2 + 3x - 4$ and $x^2 + 10x + 16$

CHALLENGE: Deena factored two polynomials and found that the polynomials had no factors in common. She concluded that the LCM of the polynomials was the same as the product of the factors. Do you agree? Explain your reasoning.

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Teaching Notes 6.8: Writing Equivalent Algebraic Fractions

Writing equivalent algebraic fractions is a prerequisite skill for adding and subtracting algebraic fractions with different denominators. Students often make errors in finding the missing factor and in multiplying.

1. Demonstrate the process for writing equivalent fractions by providing this example: $\frac{3}{4} = \frac{\quad}{20}$. Students should identify 4 and 5 as factors of 20. (There are other factors, of course, but $4 \times 5 = 20$.) To write an equivalent fraction, explain that students should multiply $\frac{3}{4}$ by $\frac{5}{5}$ with the result of $\frac{15}{20}$. Note that when they multiply by $\frac{5}{5}$ they are actually multiplying by 1, and multiplying any number by 1 does not affect the number's value.
2. Explain that the same process is used to write equivalent algebraic fractions, except that the factors may be a polynomial, number, or variable, depending on the denominator.
3. Review the information and example on the worksheet with your students. If necessary, review the steps for factoring polynomials. Be sure that students understand the steps for multiplying.

EXTRA HELP:

Use the distributive property if you multiply a binomial by a number or a variable. Use FOIL to multiply two binomials.

ANSWER KEY:

(1) $3x + 6$ **(2)** $x + 3$ **(3)** $5x - 25$ **(4)** $2x + 2$ **(5)** $x^3 + x^2$ **(6)** $x^2 + 7x + 12$

(Challenge) Because $\frac{x+2}{x+2} = 1$, the equivalent fraction must be equal to 1. The numerator must equal $x^2 + 6x + 4$.

Name _____ Date _____

WORKSHEET 6.8: WRITING EQUIVALENT ALGEBRAIC FRACTIONS

Use the steps below to write equivalent algebraic fractions. Assume that the denominators of the fractions are not equal to zero.

1. Factor the denominators.
2. Multiply the fraction by 1.

EXAMPLE

$$\frac{x}{x+1} = \frac{\qquad}{x^2+3x+2}$$

$x + 1$ cannot be factored. The factors of $x^2 + 3x + 2$ are $x + 1$ and $x + 2$.

Multiply by 1, expressed as $\frac{x+2}{x+2}$. $\quad \frac{x}{x+1} \cdot \frac{x+2}{x+2} = \frac{x^2+2x}{x^2+3x+2}$

DIRECTIONS: Write equivalent fractions.

1. $\frac{3}{x} = \frac{\qquad}{x^2+2x}$

2. $\frac{1}{x+5} = \frac{\qquad}{x^2+8x+15}$

3. $\frac{5}{x-2} = \frac{\qquad}{x^2-7x+10}$

4. $\frac{2}{x+7} = \frac{\qquad}{x^2+8x+7}$

5. $\frac{x^2}{x-2} = \frac{\qquad}{x^2-x-2}$

6. $\frac{x+4}{x+2} = \frac{\qquad}{x^2+5x+6}$

CHALLENGE: Just by looking at the problem, Samantha said that the numerator of a fraction equivalent to $\frac{x+2}{x+2} = \frac{\qquad}{x^2+6x+4}$ is $x^2 + 6x + 4$. Explain how she could have arrived at this conclusion.

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Teaching Notes 6.9: Adding and Subtracting Algebraic Fractions with Unlike Denominators

Adding and subtracting algebraic fractions requires students to be able to find the least common denominator, write equivalent fractions, and, in the case of subtraction, use the distributive property. Students might make errors in any of these operations.

1. Review the steps for adding and subtracting fractions with unlike denominators by presenting this example to your students: $\frac{7}{12} + \frac{4}{30}$.
   - Explain that students must factor the denominators to find the least common denominator (LCD). $12 = 3 \cdot 2^2, 30 = 5 \cdot 2 \cdot 3$ The LCD is $5 \cdot 2^2 \cdot 3 = 60$.
   - Point out that students must now write equivalent fractions, add, and simplify, if possible. $\frac{7}{12} \cdot \frac{5}{5} + \frac{4}{30} \cdot \frac{2}{2} = \frac{35}{60} + \frac{8}{60} = \frac{43}{60}$
   - Note that the same steps are followed for subtraction.
2. Emphasize that these same steps apply to addition and subtraction of algebraic fractions.
3. Review the information and example on the worksheet with your students. Go through the steps of the example in detail and provide additional examples if necessary. You may find it helpful to review 6.8: "Writing Equivalent Algebraic Fractions."

EXTRA HELP:

When subtracting, be sure to use the distributive property to distribute the negative sign to each term of the second fraction.

ANSWER KEY:

(1) $\frac{17x}{36}$ **(2)** $\frac{-1}{x(x+1)}$ **(3)** $\frac{2x+3}{x^2}$ **(4)** $\frac{4(x-6)}{(x+3)(x-3)}$ **(5)** $\frac{7x}{18}$

(Challenge) Shawn is incorrect. The common denominator is $(x+1)^2$. He probably forgot to write an equivalent fraction. The correct answer is $\frac{3x+8}{(x+1)^2}$.

Name _____ Date _____

WORKSHEET 6.9: ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS WITH UNLIKE DENOMINATORS

Follow the steps below to add or subtract algebraic fractions with unlike denominators. Assume that the denominators of the fractions are not equal to zero.

1. Find the least common denominator (select the largest exponent of every factor).
2. Write equivalent fractions.
3. Add or subtract.
4. Simplify, if possible.

EXAMPLE

$$\frac{2}{x+3} - \frac{1}{x+1}$$

The least common denominator is $(x+3)(x+1)$ because the factors are $(x+3)$ and $(x+1)$. The largest exponent of each factor is 1.
Write equivalent fractions and subtract.

$$\frac{2}{x+3} \cdot \frac{x+1}{x+1} - \frac{1}{x+1} \cdot \frac{x+3}{x+3} = \frac{2x+2-x-3}{(x+1)(x+3)} = \frac{x-1}{(x+1)(x+3)}$$

This fraction cannot be simplified.

DIRECTIONS: Find each sum or difference.

1. $\frac{x}{4} + \frac{2x}{9}$

2. $\frac{1}{x+1} - \frac{1}{x}$

3. $\frac{2}{x} + \frac{3}{x^2}$

4. $\frac{6}{x+3} - \frac{2}{x-3}$

5. $\frac{3x-2}{6} - \frac{x-3}{9}$

CHALLENGE: Shawn found the sum of $\frac{3}{x+1}$ and $\frac{5}{(x+1)^2}$. He added the numerators and found the sum to be $\frac{8}{(x+1)^2}$. Do you agree with his answer? Explain your reasoning.

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Teaching Notes 6.10: Multiplying and Dividing Algebraic Fractions

To multiply and divide algebraic fractions correctly, students must be competent at factoring, simplifying, and, in the case of division, using the reciprocal. The need to use multiple skills often results in errors in computation and careless mistakes.

1. Explain that algebraic fractions are multiplied and divided according to the same rules as ordinary fractions.

2. Present the following multiplication example. $\frac{8}{15} \cdot \frac{5}{12} = \frac{\cancel{8}^{2}}{\cancel{15}_{3}} \cdot \frac{\cancel{5}^{1}}{\cancel{12}_{3}} = \frac{2}{9}$ Note the process of cancelling, which requires students to divide the numerators and denominators by common factors before actually multiplying.

3. Present the following division example. $\frac{8}{15} \div \frac{5}{12} = \frac{8}{\cancel{15}_{5}} \cdot \frac{\cancel{12}^{4}}{5} = \frac{32}{25}$ Note that the divisor must be changed to its reciprocal and the division rewritten to multiplication before cancelling can be attempted.

4. Review the information and example on the worksheet with your students. Emphasize that when the numerator and denominator are algebraic expressions, which include polynomials, factoring should be written out to avoid careless errors. Remind your students to use the reciprocal when dividing. Emphasize that they should leave their answers in factored form.

EXTRA HELP:

Always rewrite division as multiplication, using the reciprocal of the fraction immediately following the division sign.

ANSWER KEY:

(1) $2(x+4)$ **(2)** $\frac{1}{x+3}$ **(3)** $\frac{x(x+4)}{10(x+1)}$ **(4)** $\frac{x^2}{2(2x+3)}$ **(5)** $\frac{2x+1}{(x-2)(x-1)}$ **(6)** $\frac{(x-4)(x-3)}{(x-2)^2}$

(Challenge) Sienna did not use the reciprocal. The correct answer is $\frac{(x-2)^2}{(x+4)(x-4)}$.

Name _____ Date _____

WORKSHEET 6.10: MULTIPLYING AND DIVIDING ALGEBRAIC FRACTIONS

Follow the steps below to multiply and divide algebraic fractions. Assume that the denominators of the fractions are not equal to zero.

1. For division problems, rewrite the problem as multiplication by using the reciprocal of the fraction immediately following the division symbol.
2. Factor the numerators and denominators.
3. Cross out the common factors (if there are any) in the numerators with the same common factors in the denominators.
4. Express your answer in factored form.

EXAMPLE

$$\frac{x+4}{x^2+4x+3} \cdot \frac{x+1}{x+3} = \frac{x+4}{(x+3)(\cancel{x+1})} \cdot \frac{\cancel{x+1}}{x+3} = \frac{x+4}{(x+3)^2}$$

DIRECTIONS: Multiply or divide. Express your answers in factored form.

1. $\frac{2x+4}{x+3} \cdot \frac{x^2+7x+12}{x+2}$

2. $\frac{x+3}{x^2-9} \div \frac{x+3}{x-3}$

3. $\frac{3x}{5} \cdot \frac{x+4}{6x+6}$

4. $\frac{3x^2}{4} \div \frac{6x+9}{2}$

5. $\frac{2x^2+3x+1}{x^2-4} \div \frac{x^2-1}{x+2}$

6. $\frac{x^2-x-12}{x-2} \cdot \frac{x-3}{x^2+x-6}$

CHALLENGE: Sienna said that $\frac{x^2-4}{x^2-16} \div \frac{x+2}{x-2} = \frac{(x+2)^2}{(x-4)(x+4)}$. Her teacher said that her answer was wrong. Find Sienna's error and correct her work.

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Teaching Notes 6.11: Solving Proportions

To solve proportions that have variables or variable expressions in the numerator or denominator, students must multiply the whole expression by the numerator or denominator, not only one term. This can result in confusion and errors.

1. Explain that a proportion is a statement that two ratios are equal.
2. Discuss solving proportions by providing this example: $\frac{2x}{5} = \frac{3}{7}$. Explain that $2x$ and 7 are called the "extremes," and that 5 and 3 are called the "means." In any proportion, the product of the means is equal to the product of the extremes. Students should write and solve the equation $2x \cdot 7 = 5 \cdot 3$ or $14x = 15$. The solution is $x = 1\frac{1}{14}$.
3. Review the information and example on the worksheet with your students. Note the use of the distributive property in the example. Depending on the abilities of your students, you may find it helpful to review solving linear equations.

EXTRA HELP:

Always use the distributive property to find the product of a polynomial and a number or variable.

ANSWER KEY:

(1) $x = 6\frac{1}{2}$ **(2)** $x = 1\frac{7}{8}$ **(3)** $x = 5\frac{3}{4}$ **(4)** $x = -4\frac{2}{3}$

(5) $x = \frac{3}{7}$ **(6)** $x = 0$ **(7)** $x = -1\frac{3}{7}$ **(8)** $x = -3$

(Challenge) Eddie is incorrect. He did not apply the distributive property correctly. The correct equation is $2x + 2 = 3x$. The solution is $x = 2$.

Name _____ Date _____

WORKSHEET 6.11: SOLVING PROPORTIONS

Use the steps that follow to solve proportions:

1. Identify the "extremes" and the "means."
2. Write an equation stating that the product of the means is equal to the product of the extremes.
3. Solve the equation.

EXAMPLE

$\frac{x-3}{4} = \frac{1}{5}$. The extremes are $x - 3$ and 5. The means are 4 and 1.

$$5(x-3) = 1 \cdot 4 \rightarrow 5x - 15 = 4 \rightarrow 5x = 19 \rightarrow x = 3\frac{4}{5}$$

DIRECTIONS: Solve each proportion.

1. $\frac{2}{5} = \frac{3}{x+1}$

2. $\frac{x-1}{7} = \frac{1}{8}$

3. $\frac{3}{x-2} = \frac{4}{5}$

4. $\frac{x+2}{4} = \frac{2}{-3}$

5. $-\frac{3}{2} = \frac{2x}{x-1}$

6. $\frac{-2}{x-4} = \frac{2}{x+4}$

7. $\frac{10}{x+5} = -\frac{4}{x}$

8. $\frac{3x+1}{2} = \frac{4x}{3}$

CHALLENGE: Eddie solved the proportion $\frac{2}{x} = \frac{3}{x+1}$ by writing the equation $2x + 1 = 3x$. He said that $x = 1$. Is he correct? If he is incorrect, what was his mistake? What is the correct solution?

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Teaching Notes 6.12: Solving Equations That Have Fractional Coefficients

To many students, equations with fractional coefficients appear to be very confusing. Fortunately, these equations can be relatively easy to solve if students multiply both sides of the equation by the least common denominator (LCD) before attempting to solve.

1. Review the multiplication property of equality, which states that multiplying both sides of an equation by the same nonzero number does not change the value of the solution. Depending on the abilities of your students, you may wish to review 3.3: "Solving Equations by Multiplying or Dividing."
2. Explain that when an equation has fractional coefficients, students must multiply both sides of the equation by the least common denominator of the coefficients. This will eliminate the fractions and students can then solve the equation. Depending on the abilities of your students, review the process for finding the least common denominator by providing some examples such as the following: Find the LCD of 18 and 20. $18 = 3^2 \cdot 2$; $20 = 2^2 \cdot 5$ The LCD is $2^2 \cdot 3^2 \cdot 5$ or 180.
3. Review the information and example on the worksheet with your students. Be sure that your students understand the use of prime factorization for finding the least common denominator. Also be sure that they understand all of the steps for solving the equation.

EXTRA HELP:

Express the product of the least common denominator and each term in factored form. This will make it easier to cancel common factors and make computation easier.

ANSWER KEY:

(1) $x = 2$ **(2)** $x = -15$ **(3)** $x = 13$ **(4)** $x = \frac{1}{7}$ **(5)** $x = 3\frac{1}{2}$

(Challenge) Yes, provided their work is correct. Both will find that $x = \frac{1}{2}$.

Name _____ Date _____

WORKSHEET 6.12: SOLVING EQUATIONS THAT HAVE FRACTIONAL COEFFICIENTS

Follow the steps below to solve equations with fractional coefficients:

1. Find the least common denominator (LCD) of the fractions.
   - Express each denominator using prime factorization with exponents.
   - Find the product of the prime numbers selecting the largest exponent of each prime number.
2. Multiply both sides of the fraction by the least common denominator.
3. Solve the equation.

EXAMPLE

$\frac{x+1}{4}+\frac{5}{6}=\frac{9}{2}$. Solve for x.

Because $4=2^2$, $6=2^1\cdot 3^1$, and $2=2^1$, the LCD is $2^2\cdot 3$ or 12. Multiply both sides of the equation by 12.

$$12\left[\frac{x+1}{4}+\frac{5}{6}=\frac{9}{2}\right]\rightarrow \frac{\overset{3}{\cancel{12}}(x+1)}{\underset{1}{\cancel{4}}}+\frac{\overset{2}{\cancel{12}}\cdot 5}{\underset{1}{\cancel{6}}}=\frac{\overset{6}{\cancel{12}}\cdot 9}{\underset{1}{\cancel{2}}}\rightarrow 3x+3+10=54\rightarrow 3x+13=54\rightarrow$$

$$3x=41\rightarrow x=13\frac{2}{3}$$

DIRECTIONS: Solve each equation.

1. $\frac{x}{6}+\frac{x}{4}=\frac{5}{6}$

2. $\frac{x}{3}-\frac{x}{4}=-\frac{5}{4}$

3. $\frac{x}{2}-\frac{x+5}{3}=\frac{1}{2}$

4. $\frac{3x}{5}-\frac{x}{4}=\frac{1}{20}$

5. $\frac{3}{8}+\frac{x+2}{4}=\frac{x}{2}$

CHALLENGE: Denny solved $\frac{x}{4}+\frac{x}{12}=\frac{1}{6}$ by multiplying the equation by 12, the least common denominator. Tara found a common denominator, 24, and multiplied the equation by 24. Will they get the same answer? Explain your reasoning.

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Teaching Notes 6.13: Solving Fractional Equations

Solving fractional equations involves not only finding the least common denominator (LCD) but also checking if the solution satisfies the original equation. A common mistake occurs when students find a solution to a fractional equation that makes the denominator of the original equation equal to zero and they include that solution as a solution to the equation.

1. Explain that a fractional equation is an equation that has a variable in the denominator of one or more terms.
2. Explain that solving a fractional equation involves the same steps as solving equations with fractional coefficients. Students must find the least common denominator and multiply both sides of the equation by the LCD to obtain an equivalent equation that has no fractions. Once the equation is solved, students must check their solution by considering if there are any restrictions on the variable in the original equation.
3. Offer this example: $\frac{3y-6}{y-2}-3=0$. Demonstrate that multiplying both sides by the least common denominator $(y-2)$ results in the equation $3y-6-3y+6=0$, which is true for all real numbers. Because the denominator of the original equation cannot equal 0, $y \neq 2$. The solution must be modified so that the solution to the original fractional equation is all real numbers except 2.
4. Review the information and example on the worksheet with your students. Make sure that your students understand the steps for solving equations. If students find a solution, they must be sure that it will not make the denominator of any term in the original equation equal to zero. Note that when they obtain a false statement, there is no solution.

EXTRA HELP:

Multiply every term of the equation by the LCD.

ANSWER KEY:

(1) $n=2$ **(2)** $x=6$ **(3)** No solution **(4)** All real numbers except 2. **(5)** $x=9$

(Challenge) Trish is correct. The solutions are $x=3$ and $x=2$. The answers were obtained by solving the fractional equation.

Name _____ Date _____

WORKSHEET 6.13: SOLVING FRACTIONAL EQUATIONS

Use the steps that follow to solve fractional equations:

1. Multiply both sides of the equation by the least common denominator (LCD) to obtain an equivalent equation that has no fractions.
2. Follow the steps for solving equations to find your solution.
3. Check the original problem to see if there are any restrictions on the variable. (Remember, the denominator of a fraction cannot equal 0.) Modify your solution, if necessary.

EXAMPLE

Solve for x. $\dfrac{4x-4}{x-1}-4=0$

$$x-1\left[\frac{4x-4}{x-1}-4=0\right]\rightarrow 4x-4-4(x-1)=0\rightarrow 4x-4-4x+4=0\rightarrow 0=0$$

Although $0=0$ is true for all real numbers, the solution to the original equation is all real numbers except 1 because the denominator cannot equal 0.

DIRECTIONS: Solve and check. If there is no solution, write "no solution."

1. $\dfrac{3}{4n}+\dfrac{1}{2n}=\dfrac{5}{8}$

2. $\dfrac{2}{x}+\dfrac{x}{x+3}=1$

3. $\dfrac{2y-5}{8y-5}-\dfrac{1}{4}=0$

4. $\dfrac{3y-6}{y-2}=3$

5. $\dfrac{3}{x}-\dfrac{1}{4}=\dfrac{1}{12}$

CHALLENGE: Trish said that $\dfrac{x+1}{2x-2}-\dfrac{3}{x}=0$ has two solutions. Do you agree? Explain your answer.

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SECTION 7

Irrational and Complex Numbers

Teaching Notes 7.1: Simplifying Radicals

To simplify radicals, students must identify perfect square factors, write a factorization, and take the square root of each factor. Lack of mastery of any of these skills leads to errors in simplification.

1. Explain to your students that a radical is the indicated root of a quantity.
2. Explain that the opposite of squaring a number is finding its square root. For example, $10^2 = 100$, therefore $\sqrt{10^2} = \sqrt{100} = 10$. Although both $10^2 = 100$ and $(-10)^2 = 100$, the principal square root is the positive value, which is the value that is primarily used. The expression $\sqrt{100}$ is called a "radical," and 100, the number inside the radical symbol, is called the "radicand."
3. Note that if the radicand is a perfect square, simplifying the radical is easy, provided students are able to identify perfect squares. For example, because 36 is a perfect square, $\sqrt{36} = 6$.
4. Explain that to simplify a radical when the radicand is not a perfect square, the radicand should be factored, if possible. No factor should have a perfect square (other than 1) as one of its factors. Note that there are several ways to begin the factoring process but the result will always be the same. Emphasize that radicals are in simplest form if 1 is the only perfect square factor.
5. Review the information and example on the worksheet with your students. Note that in the example $\sqrt{108}$ is simplified two ways. The first uses the largest perfect square factor and the second uses other perfect square factors.

EXTRA HELP:

Try to find the largest perfect square factor and use this to write the factorization.

ANSWER KEY:

(1) $5\sqrt{6}$ **(2)** $3\sqrt{7}$ **(3)** $4\sqrt{6}$ **(4)** $10\sqrt{2}$ **(5)** $2\sqrt{19}$ **(6)** $4\sqrt{3}$

(7) Cannot be simplified **(8)** $6\sqrt{10}$ **(9)** Cannot be simplified **(10)** $3\sqrt{6}$ **(11)** $5\sqrt{22}$ **(12)** $6\sqrt{7}$

(Challenge) Suresh's answer is partially correct. He has to simplify $\sqrt{20}$ further because the factors of 20 are 4 and 5. 4 is a perfect square. $\sqrt{80} = 4\sqrt{5}$

Name _____ Date _____

WORKSHEET 7.1: SIMPLIFYING RADICALS

A radical is in simplest form if there are no perfect square factors other than 1 in the radicand. Follow the steps below to simplify radicals:

1. Factor the radicand as a product of the largest perfect square and another number. (The radicand is the number inside the radical symbol.) Examples of perfect squares include 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
2. If you did not select the largest perfect square factor, determine if the other number has a perfect square factor. If it does, factor the number.
3. Continue factoring until there are no numbers in the radicand that have a perfect square (other than 1) as a factor.
4. Simplify your answer.

EXAMPLE

Simplify $\sqrt{108}$.

$$\sqrt{108} = \sqrt{36} \cdot \sqrt{3} = 6\sqrt{3}$$

or

$$\sqrt{108} = \sqrt{4} \cdot \sqrt{27} = \sqrt{4} \cdot \sqrt{9} \cdot \sqrt{3} = 2 \cdot 3 \cdot \sqrt{3} = 6\sqrt{3}$$

DIRECTIONS: Simplify each radical. If the radical cannot be simplified, write "cannot be simplified."

1. $\sqrt{150}$	**2.** $\sqrt{63}$	**3.** $\sqrt{96}$	**4.** $\sqrt{200}$
5. $\sqrt{76}$	**6.** $\sqrt{48}$	**7.** $\sqrt{55}$	**8.** $\sqrt{360}$
9. $\sqrt{42}$	**10.** $\sqrt{54}$	**11.** $\sqrt{550}$	**12.** $\sqrt{252}$

CHALLENGE: Suresh simplified $\sqrt{80}$ and said that it equals $2\sqrt{20}$. Is he correct? Explain your answer.

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Teaching Notes 7.2: Multiplying Radicals

To multiply radicals, students must use the product property of square roots. This property not only allows students to multiply radicals, but it also allows them to simplify radicals.

1. Explain that multiplying radicals is similar to multiplying integers. The product property of square roots states that $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$, where x and y are real numbers that are greater than or equal to 0. The product of two square roots is found by multiplying the radicands. For example, $\sqrt{5} \cdot \sqrt{10} = \sqrt{50}$.
2. Explain that an answer is in simplest form when there are no perfect square factors other than 1 in the radicand. Ask your students if $\sqrt{50}$ is in simplest form. The answer, of course, is no.
3. Review how to simplify radicals by presenting this example: Simplify $\sqrt{50}$ by writing $\sqrt{50}$ as the product of $\sqrt{25}$ and $\sqrt{2}$. Because $\sqrt{25} = 5$, $\sqrt{50}$ expressed in simplest form is $5\sqrt{2}$. Depending on the abilities of your students, you may wish to review 7.1: "Simplifying Radicals."
4. Review the information and the example on the worksheet with your students. Note that it is best to multiply the coefficients last.

EXTRA HELP:

Knowing your multiplication facts is a prerequisite skill for multiplying radicals.

ANSWER KEY:

(1) $2\sqrt{5}$ **(2)** $2\sqrt{14}$ **(3)** $18\sqrt{2}$ **(4)** $3\sqrt{5}$ **(5)** $2\sqrt{35}$ **(6)** $9\sqrt{6}$ **(7)** $-4\sqrt{15}$ **(8)** $-20\sqrt{5}$

(9) $-60\sqrt{6}$ **(10)** 24 **(11)** $3\sqrt{30}$ **(12)** -32

(Challenge) His method will work; however, Bobby needs to simplify his product, $2\sqrt{12}$, as $4\sqrt{3}$, which is the correct answer.

Name _____ Date _____

WORKSHEET 7.2: MULTIPLYING RADICALS

Multiplying radicals requires the use of the product property of square roots that states the square root of a product equals the product of the square root of the factors. $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$, where x and y are real numbers that are greater than or equal to 0. Follow the steps below:

1. Multiply the radicands.
2. Simplify the radicals.
3. Multiply the coefficients.

EXAMPLE

$4\sqrt{3} \cdot \sqrt{6}$

$4\sqrt{3} \cdot \sqrt{6} = 4\sqrt{18}$

$4\sqrt{18}$ can be simplified as $4\sqrt{9} \cdot \sqrt{2}$.

$4 \cdot 3\sqrt{2} = 12\sqrt{2}$

DIRECTIONS: Simplify each expression.

1. $\sqrt{4} \cdot \sqrt{5}$ **2.** $\sqrt{7} \cdot \sqrt{8}$ **3.** $3\sqrt{6} \cdot 2\sqrt{3}$

4. $\sqrt{15} \cdot \sqrt{3}$ **5.** $2\sqrt{5} \cdot \sqrt{7}$ **6.** $3\sqrt{3} \cdot \sqrt{18}$

7. $-4\sqrt{3} \cdot \sqrt{5}$ **8.** $-5\sqrt{10} \cdot \sqrt{8}$ **9.** $-3\sqrt{10} \cdot 4\sqrt{15}$

10. $6\sqrt{8} \cdot \sqrt{2}$ **11.** $3\sqrt{10} \cdot \sqrt{3}$ **12.** $8\sqrt{2} \cdot -2\sqrt{2}$

CHALLENGE: Bobby said that he found a faster way to multiply radicals by simplifying a radical before he multiplied. For example, when Bobby was solving the problem $\sqrt{8} \cdot \sqrt{6}$, he first expressed $\sqrt{8}$ as $2\sqrt{2}$ and then multiplied $2\sqrt{2}$ and $\sqrt{6}$, which equals $2\sqrt{12}$. Do you agree with his method? Will it always work? Explain.

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Teaching Notes 7.3: Rationalizing the Denominator

To express radicals in simplest form, no radicals should appear in the denominator of a fraction. Students often make mistakes in the procedure for rationalizing the denominator.

1. Explain that a radical expression is in simplest form when there are no perfect square factors other than 1 in the radicand and no radicals appear in the denominator of a fraction. When a radical appears in the denominator of a fraction, students must rationalize the denominator in order to express the radical in simplest terms.
2. Demonstrate how to rationalize the denominator using the example $\frac{2}{\sqrt{3}}$. Explain that to eliminate the radical in the denominator, students must multiply the fraction by $\frac{\sqrt{3}}{\sqrt{3}}$. Note that this fraction is equivalent to 1; therefore, they are not changing the value of the original fraction. Explain that $\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{9}}$, which is simplified as $\frac{2\sqrt{3}}{3}$.
3. Review the information and example on the worksheet with your students.

EXTRA HELP:

You can have a radical in the numerator of your answer but you cannot have a radical in the denominator.

ANSWER KEY:

(1) $\frac{5\sqrt{7}}{7}$ (2) $\sqrt{2}$ (3) $\frac{5\sqrt{6}}{3}$ (4) $\frac{\sqrt{10}}{5}$ (5) $\frac{\sqrt{15}}{5}$ (6) $\frac{\sqrt{21}}{7}$ (7) $\frac{\sqrt{10}}{4}$ (8) $-\sqrt{3}$ (9) $\frac{2\sqrt{6}}{3}$

(10) $\frac{\sqrt{33}}{11}$ (11) $\frac{\sqrt{2}}{2}$ (12) $\sqrt{3}$

(Challenge) If the radicand in the denominator is a perfect square, it is not necessary to rationalize the denominator. It is only necessary to find the square root. For example, $\frac{5}{\sqrt{9}} = \frac{5}{3}$.

Name _____ Date _____

WORKSHEET 7.3: RATIONALIZING THE DENOMINATOR

Sometimes writing an expression in simplest form requires rationalizing the denominator. Follow the steps below to rationalize the denominator:

1. Multiply the numerator and denominator by the radical in the denominator.
2. Multiply fractions.
3. Simplify the radical.
4. Simplify the fraction.

EXAMPLE

$$\frac{3}{\sqrt{12}}$$

$$\frac{3}{\sqrt{12}} \cdot \frac{\sqrt{12}}{\sqrt{12}} = \frac{3\sqrt{12}}{\sqrt{144}} = \frac{3\sqrt{12}}{12} = \frac{3\sqrt{4}\sqrt{3}}{12} = \frac{3 \cdot 2\sqrt{3}}{12} = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$$

DIRECTIONS: Simplify each expression.

1. $\frac{5}{\sqrt{7}}$ **2.** $\frac{4}{\sqrt{8}}$ **3.** $\frac{10}{\sqrt{6}}$ **4.** $\frac{2}{\sqrt{10}}$

5. $\frac{\sqrt{3}}{\sqrt{5}}$ **6.** $\frac{\sqrt{3}}{\sqrt{7}}$ **7.** $\frac{\sqrt{5}}{\sqrt{8}}$ **8.** $\frac{-\sqrt{9}}{\sqrt{3}}$

9. $\frac{\sqrt{8}}{\sqrt{3}}$ **10.** $\frac{\sqrt{3}}{\sqrt{11}}$ **11.** $\frac{\sqrt{5}}{\sqrt{10}}$ **12.** $\frac{\sqrt{12}}{2}$

CHALLENGE: Give an example of an expression that contains a radical in the denominator that can be simplified without rationalizing the denominator.

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Teaching Notes 7.4: Dividing Radicals

To divide radicals, students must use the quotient property of square roots. As students apply this property, they must be sure there is no radical in the denominator.

1. Explain that dividing radicals is similar to dividing integers. The quotient property of square roots states that $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ where x is a real number greater than or equal to 0 and y is a real number greater than 0. The quotient of two square roots is found by dividing the radicands. For example, $\frac{\sqrt{30}}{\sqrt{2}} = \sqrt{\frac{30}{2}} = \sqrt{15}$.

2. Review the information and example on the worksheet with your students. Make sure that your students understand the quotient property of square roots. Explain that a radical is in simplest form when there are no perfect square factors other than 1 in the radicand and there are no radicals in the denominator. If necessary, present this example: $\sqrt{\frac{2}{10}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{1}}{\sqrt{5}} = \frac{1}{\sqrt{5}}$. Emphasize that this expression must be simplified by rationalizing the denominator. $\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{25}} = \frac{\sqrt{5}}{5}$ You may find it helpful to review 7.3: "Rationalizing the Denominator."

EXTRA HELP:

You can only divide one radicand by another radicand.

ANSWER KEY:

(1) $\frac{\sqrt{5}}{5}$ **(2)** $\frac{\sqrt{70}}{10}$ **(3)** $4\sqrt{3}$ **(4)** 2 **(5)** $\frac{3\sqrt{3}}{2}$ **(6)** $\frac{3\sqrt{10}}{10}$ **(7)** $\sqrt{5}$ **(8)** $\frac{\sqrt{30}}{3}$

(Challenge) Her work is correct but she needs to continue to simplify $\sqrt{4}$. The correct answer is 2.

Name _____ Date _____

WORKSHEET 7.4: DIVIDING RADICALS

To divide radicals you must use the quotient property of square roots. This property states that the square root of a quotient equals the quotient of the square roots. $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$, where x is a real number greater than or equal to 0 and y is a real number that is greater than 0. Follow the steps below to divide radicals:

1. If one radical is a factor of the other, divide the radicals. If the radicals have a common factor, simplify.
2. Simplify the radical expression.
   - Radicands should have no common perfect square factor other than 1.
   - No radicals should be in the denominator.

EXAMPLE

$$\frac{3\sqrt{12}}{\sqrt{10}}$$

Simplify the radicals because $\sqrt{10}$ is not a factor of $\sqrt{12}$. $\frac{3\sqrt{12}}{\sqrt{10}} = 3\sqrt{\frac{12}{10}} = 3\sqrt{\frac{6}{5}} = \frac{3\sqrt{6}}{\sqrt{5}}$

Simplify the radical expression. $\frac{3\sqrt{6}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{30}}{\sqrt{25}} = \frac{3\sqrt{30}}{5}$

DIRECTIONS: Simplify each expression.

1. $\frac{\sqrt{3}}{\sqrt{15}}$
2. $\frac{\sqrt{14}}{\sqrt{20}}$
3. $\frac{4\sqrt{6}}{\sqrt{2}}$
4. $\frac{\sqrt{20}}{\sqrt{5}}$
5. $\frac{3\sqrt{6}}{\sqrt{8}}$
6. $\frac{\sqrt{27}}{\sqrt{30}}$
7. $\frac{\sqrt{15}}{\sqrt{3}}$
8. $\frac{\sqrt{40}}{\sqrt{12}}$

CHALLENGE: Mary simplified $\sqrt{\frac{64}{16}}$ as $\sqrt{4}$. Is she correct? If not, explain why not.

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Teaching Notes 7.5: Adding and Subtracting Radicals

Only radicals with like radicands can be added or subtracted. A common mistake students make is to add or subtract all radicands.

1. Explain that to add or subtract radicals, the radicands (the numbers under the square root symbol) must be the same.
2. Ask your students to add and simplify $\sqrt{9} + \sqrt{16}$. The correct answer is $3 + 4 = 7$. Compare this with the common error of adding $\sqrt{9} + \sqrt{16}$ with a result of $\sqrt{25} = 5$.
3. Now ask your students to subtract and simplify $\sqrt{100} - \sqrt{64}$. The correct answer is $10 - 8 = 2$. Compare this with the common error of subtracting $\sqrt{100} - \sqrt{64}$ with a result of $\sqrt{36} = 6$.
4. Review the information and examples on the worksheet with your students. Sometimes students look only at the radicand and assume that because the radicands vary, the radicals cannot be added or subtracted. Encourage your students to simplify the radicals (if possible) and then determine if the radicals may be added or subtracted. Note that if there is no number in front of the radical sign, the number is assumed to be 1. For example, $\sqrt{7} + \sqrt{7} = 2\sqrt{7}$. Also note that not all radicals can be added or subtracted, even when the radical is simplified.

EXTRA HELP:

Although all radicands can be multiplied, only like radicands can be added or subtracted.

ANSWER KEY:

(1) $-\sqrt{10}$	**(2)** $2\sqrt{2} + 2\sqrt{5}$	**(3)** $4 - 6\sqrt{2}$	**(4)** $\sqrt{5} + \sqrt{10}$	**(5)** 13
(6) $\sqrt{15} - 3\sqrt{10}$	**(7)** $5\sqrt{7}$	**(8)** $2\sqrt{3} - \sqrt{10}$	**(9)** $\sqrt{15} + 2\sqrt{5}$	**(10)** $-13\sqrt{3}$
(11) $-6\sqrt{2} - 10\sqrt{10}$	**(12)** $\sqrt{5}$			
(Challenge) Agree. Explanations may vary. A possible explanation is $2 + 2 = 4$.				

Name _____ Date _____

WORKSHEET 7.5: ADDING AND SUBTRACTING RADICALS

Follow the steps below to add or subtract radicals:

1. Only radicals with the same radicands can be added or subtracted.
2. If the radicals are different, try to simplify one or both of the radicals.
3. Once the radicals are simplified, add or subtract the numbers directly preceding the radicals that are the same.

EXAMPLES

$$\sqrt{5}+\sqrt{45}=\sqrt{5}+\sqrt{9}\cdot\sqrt{5}=\sqrt{5}+3\sqrt{5}=4\sqrt{5}$$

$$\sqrt{48}-\sqrt{27}=\sqrt{16}\cdot\sqrt{3}-\sqrt{9}\cdot\sqrt{3}=4\sqrt{3}-3\sqrt{3}=\sqrt{3}$$

$$\sqrt{20}+\sqrt{32}=\sqrt{4}\cdot\sqrt{5}+\sqrt{16}\cdot\sqrt{2}=2\sqrt{5}+4\sqrt{2}$$

DIRECTIONS: Find the sums or differences.

1. $\sqrt{40}-\sqrt{90}$ **2.** $\sqrt{8}+\sqrt{20}$ **3.** $\sqrt{16}-\sqrt{72}$

4. $\sqrt{5}+\sqrt{10}$ **5.** $\sqrt{9}+\sqrt{100}$ **6.** $\sqrt{15}-\sqrt{90}$

7. $\sqrt{28}+\sqrt{63}$ **8.** $\sqrt{12}-\sqrt{10}$ **9.** $\sqrt{15}+\sqrt{20}$

10. $\sqrt{12}-5\sqrt{27}$ **11.** $-3\sqrt{8}-5\sqrt{40}$ **12.** $\sqrt{45}-\sqrt{20}$

CHALLENGE: $\sqrt{4}+\sqrt{4}=\sqrt{16}$. Do you agree or disagree? Explain your reasoning.

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Teaching Notes 7.6: Multiplying Two Binomials Containing Radicals

To multiply two binomials containing radicals, students must be familiar with multiplying radicals and multiplying binomials. Common mistakes include using FOIL incorrectly, multiplying radicals incorrectly, or simplifying radicals incorrectly.

1. Review how to multiply radicals with your students. Note that students must multiply the radicands and simplify the radical. It may be helpful to look back at 7.2: "Multiplying Radicals."
2. Review the information and example on the worksheet with your students. Emphasize how the FOIL method is used to multiply these terms. Depending on the abilities of your students, it may be necessary to review multiplying binomials by presenting examples such as $(x+3)(x-2) = x^2 - 2x + 3x - 6 = x^2 + x - 6$. Caution your students to pay close attention to negative signs. Remind them that only radicals with like radicands can be added or subtracted.

EXTRA HELP:

You may simplify radicals before you multiply. However, it may still be necessary to check that all radicals in the product are in simplest form.

ANSWER KEY:

(1) $\sqrt{21} + 5\sqrt{7} + 3\sqrt{3} + 15$ **(2)** -2 **(3)** $5 + 2\sqrt{6}$ **(4)** $2\sqrt{10} + 2\sqrt{5} - 2\sqrt{6} - 2\sqrt{3}$

(5) $4\sqrt{3} - 2\sqrt{6} - \sqrt{30} + \sqrt{15}$ **(6)** $8 - 4\sqrt{5} - 2\sqrt{2} + \sqrt{10}$

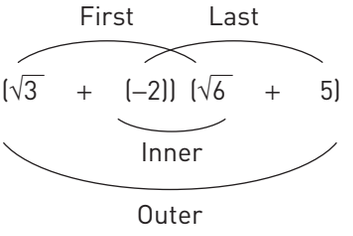
(Challenge) Joelle's teacher is correct. $(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3}) = -1$.

Name _____ Date _____

WORKSHEET 7.6: MULTIPLYING TWO BINOMIALS CONTAINING RADICALS

Follow the steps below to multiply two binomials containing radicals:

1. Multiply the first terms.
2. Multiply the outer terms.
3. Multiply the inner terms.
4. Multiply the last terms.
5. Add or subtract and simplify if necessary.



EXAMPLE

$$(\sqrt{3}-2)(\sqrt{6}+5)$$

$$\sqrt{3}\cdot\sqrt{6}=\sqrt{18}$$

$$\sqrt{3}\cdot 5=5\sqrt{3}$$

$$-2\cdot\sqrt{6}=-2\sqrt{6}$$

$$-2\cdot 5=-10$$

$$\sqrt{18}+5\sqrt{3}-2\sqrt{6}-10=\sqrt{9}\sqrt{2}+5\sqrt{3}-2\sqrt{6}-10=3\sqrt{2}+5\sqrt{3}-2\sqrt{6}-10$$

DIRECTIONS: Simplify each expression.

1. $(\sqrt{7}+3)(\sqrt{3}+5)$ **2.** $(3+\sqrt{11})(3-\sqrt{11})$ **3.** $(\sqrt{3}+\sqrt{2})(\sqrt{3}+\sqrt{2})$

4. $(\sqrt{5}-\sqrt{3})(\sqrt{8}+2)$ **5.** $(\sqrt{8}-\sqrt{5})(\sqrt{6}-\sqrt{3})$ **6.** $(4-\sqrt{2})(2-\sqrt{5})$

CHALLENGE: Joelle's teacher told her that multiplying two binomials that contain radicals does not necessarily mean the product will contain a radical. Do you agree with this statement? Provide an example if you agree or a counterexample if you disagree.

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Teaching Notes 7.7: Using Conjugates to Simplify Radical Expressions

Using conjugates to eliminate radicals in the denominator of a rational expression requires students to find the conjugate and multiply expressions that contain one or more terms. Errors are made when students do not eliminate the radical in the denominator or when they make mistakes in multiplication.

1. Explain that the denominator of a fraction may contain the sum or difference of a number and radical. If this is the case, students will need to use conjugates to eliminate the radical or radicals in the denominator.
2. Explain that students may eliminate a term in binomial multiplication if one operation is the opposite of the other. Offer this example: $(x + 3)(x - 3) = x^2 - 3x + 3x - 9 = x^2 - 9$. Explain that the same principle applies to binomial multiplication with opposite operations with radicals. Provide this example:

$$(x - \sqrt{3})(x + \sqrt{3}) = x^2 - x\sqrt{3} + x\sqrt{3} - \sqrt{9} = x^2 - x\sqrt{3} + x\sqrt{3} - 3 = x^2 - 3$$

3. Explain that expressions such as $x - \sqrt{3}$ and $x + \sqrt{3}$ are called "conjugates." Once conjugates are multiplied, the product contains no radicals.
4. Review the information and example on the worksheet with your students. Note the use of the distributive property to multiply the numerators. Remind students that a simplified expression does not have a radical in the denominator.

EXTRA HELP:

Conjugates are used to simplify radical expressions only when the denominator is the sum or difference of a radical expression.

ANSWER KEY:

(1) $\frac{8 - 2\sqrt{3}}{13}$ **(2)** $\frac{35 + 5\sqrt{5}}{44}$ **(3)** $\frac{50 - 10\sqrt{2}}{23}$ **(4)** $\frac{-5\sqrt{7} - 25}{6}$ **(5)** $\frac{-6\sqrt{2} + 20}{41}$ **(6)** $\frac{-12 - 21\sqrt{3}}{131}$

(Challenge) Because conjugates are binomials that differ only by the sign of one term, a radical and its opposite are added. The sum of the radicals is equal to 0.

Name _____ Date _____

WORKSHEET 7.7: USING CONJUGATES TO SIMPLIFY RADICAL EXPRESSIONS

If the denominator of a radical expression contains the sum or difference of a number and a radical, use a conjugate to simplify the expression. A conjugate is an expression with the opposite operation of the original expression. For example, the expressions $5 + \sqrt{3}$ and $5 - \sqrt{3}$ are conjugates. Follow the steps below to use conjugates to simplify radical expressions:

1. Identify the conjugate of the expression in the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Simplify the product.

EXAMPLE

Simplify $\frac{3}{2+\sqrt{3}}$.

$2 - \sqrt{3}$ is the conjugate of the expression in the denominator.

$$\frac{3}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}}$$

$$\frac{3(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{6-3\sqrt{3}}{4-2\sqrt{3}+2\sqrt{3}-\sqrt{9}} = \frac{6-3\sqrt{3}}{4-3}$$

$$\frac{6-3\sqrt{3}}{1} = 6-3\sqrt{3}$$

DIRECTIONS: Simplify each expression.

1. $\frac{2}{4+\sqrt{3}}$
2. $\frac{5}{7-\sqrt{5}}$
3. $\frac{10}{5+\sqrt{2}}$
4. $\frac{15}{\sqrt{7}-5}$
5. $\frac{4}{3\sqrt{2}-10}$
6. $\frac{3}{4-7\sqrt{3}}$

CHALLENGE: Explain why multiplying by a conjugate eliminates the radical sign.

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Teaching Notes 7.8: Simplifying Square Roots of Negative Numbers

The concept of an imaginary number—a number that involves the imaginary unit i—is abstract. Students often have trouble rewriting expressions by factoring out the $\sqrt{-1}$ and simplifying expressions using i.

1. Explain that there are two types of numbers—real numbers and imaginary numbers. Examples of real numbers include the counting numbers, zero, negative integers, rational numbers, and irrational numbers. You might want to offer some examples, such as 3, -7, $\frac{2}{5}$, $0.\overline{3}$, π, and so on.
2. Explain that the square root of a negative number is an imaginary number. There is no real number squared whose product is a negative number. Note that an imaginary number may be simplified in a manner that is similar to simplifying radicals. You may find it helpful to review 7.1: "Simplifying Radicals."
3. Review the information and example on the worksheet with your students. Make sure that your students understand the product property of square roots. $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$ where x and y are real numbers that are greater than or equal to 0. Explain that when finding the square root of a negative number, the $\sqrt{-1}$ must be factored first, then the radical should be simplified, if possible. When numbers precede the radical, the numbers can be multiplied.

EXTRA HELP:

It is customary to write i directly before the radical symbol. For example, $i \cdot 4\sqrt{3}$ should be written as $4i\sqrt{3}$.

ANSWER KEY:

(1) $4i$ **(2)** $8i\sqrt{5}$ **(3)** $3i\sqrt{3}$ **(4)** $2i\sqrt{3}$ **(5)** $2i\sqrt{30}$ **(6)** $9i\sqrt{2}$ **(7)** $i\sqrt{10}$ **(8)** $12i\sqrt{6}$

(9) $6i\sqrt{6}$ **(10)** $-20i\sqrt{2}$

(Challenge) Marta is incorrect. She can further simplify $8i\sqrt{8}$ as $16i\sqrt{2}$.

Name _____ Date _____

WORKSHEET 7.8: SIMPLIFYING SQUARE ROOTS OF NEGATIVE NUMBERS

The square root of a negative number can be simplified by using the imaginary unit i, which is equal to $\sqrt{-1}$. Follow the steps below to simplify the square roots of negative numbers:

1. Factor the radicand by finding the product of $\sqrt{-1}$ and the other radicand.
2. Replace $\sqrt{-1}$ by i.
3. Simplify the radical.
4. Multiply the numbers directly preceding the radicals, if applicable.

EXAMPLE

Simplify $7\sqrt{-48}$.

$$7\sqrt{-48} = 7\sqrt{-1} \cdot \sqrt{48} = 7i\sqrt{48} = 7i\sqrt{16} \cdot \sqrt{3} = 7i \cdot 4\sqrt{3} = 28i\sqrt{3}$$

DIRECTIONS: Simplify each expression.

1. $\sqrt{-16}$

2. $4\sqrt{-20}$

3. $\sqrt{-27}$

4. $\sqrt{-12}$

5. $2\sqrt{-30}$

6. $3\sqrt{-18}$

7. $\sqrt{-10}$

8. $4\sqrt{-54}$

9. $3\sqrt{-24}$

10. $-4\sqrt{-50}$

CHALLENGE: Marta expressed $\sqrt{-512}$ in simplified form as $8i\sqrt{8}$. Was she correct? Explain your answer.

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Teaching Notes 7.9: Multiplying Imaginary Numbers

Multiplying imaginary numbers causes two major problems for students. The product property of square roots does not apply if both radicands are negative, and the powers of i are easily confused.

1. Review how to simplify the square roots of negative numbers by presenting this example: $\sqrt{-10} = \sqrt{-1} \cdot \sqrt{10} = i\sqrt{10}$. Depending on the abilities of your students, you might find it helpful to refer to 7.8: "Simplifying Square Roots of Negative Numbers."
2. Review the product property of square roots, $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$, where x and y are real numbers that are greater than or equal to 0. Explain that this property applies to imaginary numbers provided x or y is negative; it does not apply if both x and y are negative. Note that it is a good strategy to factor out $\sqrt{-1}$ and replace it by i before applying the property.
3. Offer this example: $\sqrt{-10} \cdot \sqrt{-3}$. Explain that students must simplify each radical to eliminate the negative numbers in both radicands before they multiply. $\sqrt{-10} \cdot \sqrt{-3} = i\sqrt{10} \cdot i\sqrt{3} = i^2\sqrt{30}$
4. Explain that $i^2\sqrt{30}$ is not in simplest form. Because $i = \sqrt{-1}$, $i^2 = (\sqrt{-1})^2 = -1$. Thus, in simplest form $i^2\sqrt{30}$ is $-\sqrt{30}$.
5. Review the information and examples on the worksheet with your students. Discuss the examples fully, making sure your students understand all of the steps.

EXTRA HELP:

The product of two negative radicands is never a positive number. Remember to factor $\sqrt{-1}$ before you multiply.

ANSWER KEY:

(1) $6i$ **(2)** -10 **(3)** $-\sqrt{6}$ **(4)** i **(5)** $-4i$ **(6)** $-2i\sqrt{2}$ **(7)** -6 **(8)** $6i\sqrt{10}$ **(9)** $8i\sqrt{15}$

(10) $-4i\sqrt{5}$ **(11)** -9 **(12)** -20

(Challenge) Although Vicky applied the product property of square roots to the product of two negative radicands, the property does not apply to two negative radicands. She should have factored $\sqrt{-1}$ out of both radicands. The correct answer is $-2\sqrt{5}$.

Name _____ Date _____

WORKSHEET 7.9: MULTIPLYING IMAGINARY NUMBERS

To multiply imaginary numbers, follow the steps below:

1. Factor $\sqrt{-1}$ from the negative radicands.
2. Substitute $\sqrt{-1}$ with i.
3. Multiply the radicands and simplify the product.
4. Multiply the i's using the pattern $i = \sqrt{-1},\ i^2 = -1,\ i^3 = -1\sqrt{-1} = -i,\ i^4 = 1, \ldots$
5. Multiply the numbers preceding the radicals.

EXAMPLES

$$\sqrt{-2} \cdot \sqrt{-3} = \sqrt{-1} \cdot \sqrt{2} \cdot \sqrt{-1} \cdot \sqrt{3} = i\sqrt{2} \cdot i\sqrt{3} = i^2\sqrt{6} = -\sqrt{6}$$

$$6\sqrt{7} \cdot \sqrt{-8} = 6\sqrt{7} \cdot \sqrt{-1} \cdot \sqrt{8} = 6i\sqrt{56} = 6i\sqrt{4} \cdot \sqrt{14} = 6i \cdot 2\sqrt{14} = 12i\sqrt{14}$$

DIRECTIONS: Simplify each expression.

1. $\sqrt{-3} \cdot \sqrt{12}$

2. $\sqrt{-4} \cdot \sqrt{-25}$

3. $\sqrt{-2} \cdot \sqrt{-3}$

4. i^5

5. $-\sqrt{-16}$

6. $i\sqrt{-2} \cdot \sqrt{-4}$

7. $3i \cdot 2i$

8. $3\sqrt{2} \cdot \sqrt{-20}$

9. $4\sqrt{2} \cdot \sqrt{-30}$

10. $-\sqrt{20} \cdot \sqrt{-4}$

11. $\sqrt{-3} \cdot \sqrt{-27}$

12. $-4i \cdot -5i$

CHALLENGE: Vicky completed this problem: $\sqrt{-2} \cdot \sqrt{-10} = \sqrt{-2 \cdot -10} = \sqrt{20} = 2\sqrt{5}$. Her work is incorrect. What could be her mistake? Provide the correct answer.

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Teaching Notes 7.10: Simplifying Complex Numbers

Simplifying complex numbers is similar to simplifying binomials. A common error students make in simplifying complex numbers is substituting for i^2 incorrectly.

1. Explain to your students that complex numbers can be expressed in the form $x + yi$ where x and y are real numbers. Emphasize that x is called the "real" part of $x + yi$ and y is called the "imaginary" part. Examples include $3 + 2i$, $-4 - 2i$, and $6 - 7i$.
2. Review the information and examples on the worksheet with your students. Explain that operations with complex numbers are the same as operations with binomials. Because complex numbers are used in several operations, fully discuss each step, especially for multiplication and division, with your students. Be sure that your students understand the application of FOIL.
3. Emphasize that the conjugates differ only by a sign. In the example for division, $6-7i$ and $6 + 7i$ are the conjugates of the denominator.

EXTRA HELP:

If your answer contains i^2, replace i^2 with -1 and simplify your answer.

ANSWER KEY:

(1) $7 + 2i$ **(2)** $-2-3i$ **(3)** $3 - 5i$ **(4)** $-4 + 7i$ **(5)** $-5-12i$ **(6)** $2-6i$ **(7)** $1 + i$ **(8)** $\frac{18 + i}{13}$

(Challenge) Andrew is wrong. It appears that he multiplied $(3 - i)(2 - i)$. The correct solution is $5 - 2i$.

Name _____ Date _____

WORKSHEET 7.10: SIMPLIFYING COMPLEX NUMBERS

A complex number is a number of the form $x + yi$, where x and y are real numbers. x is the "real" part of $x + yi$. y is the "imaginary" part. Use the guidelines below to simplify complex numbers:

- *Addition or subtraction:* Combine the real parts, then combine the imaginary parts.
- *Multiplication:* Use FOIL. Simplify by combining the real parts, then combine the imaginary parts. Replace i^2 by -1.
- *Division:* Multiply the numerator and denominator by the conjugate of the denominator. Simplify the numerator. Simplify the denominator. Replace i^2 by -1.

EXAMPLES

Addition: $(3 + 2i) + (6 - 7i) = 9 - 5i$

Subtraction: $(3 + 2i) - (6 - 7i) = 3 + 2i - 6 + 7i = -3 + 9i$

Multiplication: $(3 + 2i)(6 - 7i) = 18 - 9i - 14i^2 = 18 - 9i - 14(-1) = 32 - 9i$

Division: $\dfrac{3 + 2i}{6 - 7i} = \dfrac{3 + 2i}{6 - 7i} \cdot \dfrac{6 + 7i}{6 + 7i} = \dfrac{18 + 33i + 14i^2}{36 - 49i^2} = \dfrac{18 + 33i + 14(-1)}{36 - 49(-1)} = \dfrac{4 + 33i}{85}$

DIRECTIONS: Simplify each expression.

1. $(3 - 6i) + (4 + 8i)$

2. $(1 - 2i) - (3 + i)$

3. $(1 - i)(4 - i)$

4. $(2 + 3i)(1 + 2i)$

5. $(2 - 3i)(2 - 3i)$

6. $(4 - 2i)(1 - i)$

7. $\dfrac{(1 + 3i)}{(2 + i)}$

8. $\dfrac{(3 - 4i)}{(2 - 3i)}$

CHALLENGE: Andrew said that $(3 - i) + (2 - i) = 6 - 5i + i^2$ or $5 - 5i$. Is he correct? If he is incorrect, explain what he might have done wrong and provide the correct answer.

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SECTION 8

Functions

Teaching Notes 8.1: Determining if a Relation Is a Function

A function is a relation if each value in the domain is paired with exactly one member of the range. Students often incorrectly believe that each member of the range must be paired with exactly one member of the domain.

1. Explain that a relation is any set of ordered pairs. A function is a special type of relation. Emphasize that each member of the domain must be assigned exactly one member of the range and each member of the range must be assigned at least one member of the domain.
2. To illustrate the meaning of exactly one, offer this example. A group of three girls — Brittany, Rachel, and Inez — represent the domain. A group of three boys — Roberto, John, and Tyrone — represent the range. Both groups go to the mall. While shopping, Brittany sees Roberto, Rachel sees John, and Inez sees Tyrone. This is an example of each member of the domain being paired with exactly one member of the range. Point out that it does not matter if, while shopping, John sees Brittany or Inez, or that Tyrone sees Brittany or Rachel. Each member of the domain was paired with exactly one member of the range.
3. Review the information and examples on the worksheet with your students. Discuss the definition of a function and emphasize that "exactly one" means "one and only one" (not two or more). "At least one" means "one or more." Make sure that your students understand why or why not a relation is a function.

EXTRA HELP:

Always consider the x-values first. Be sure that each value of x is paired with exactly one y-value. Looking at the y-values first leads to confusion.

ANSWER KEY:

(1) Function

(2) Not a function (1 is paired with 1, 2, 3, and 4.)

(3) Function

(4) Not a function (2 is paired with 4 and −4.)

(5) Not a function (3 is paired with 2 and −2, and 4 is paired with 15 and −15.)

(6) Function

(7) Not a function (2 is paired with 0, 4, and −4.)

(8) Function

(9) Not a function (−4 is paired with 2 and 4, and 0 is paired with 3 and 5.)

(Challenge) Answers may vary. (1, 5), (2, 6), (3, 7), (4, 8) is one example.

Name _____ Date _____

WORKSHEET 8.1: DETERMINING IF A RELATION IS A FUNCTION

Follow the steps below to determine if a relation is a function:

1. Find the *x*-values and *y*-values of the ordered pairs.
   - The *x*-values are the first numbers in ordered pairs. They are called the "domain."
   - The *y*-values are the second numbers in ordered pairs. They are called the "range."
2. If each value of *x* is paired with exactly one value of *y*, the relation is a function.
3. If one value of *x* is paired with two or more different values of *y*, the relation is not a function.

EXAMPLES

(1, 2), (2, 4), (3, 6), (4, 8) is a function. Each value of *x* is paired with exactly one value of *y*.
(0, 0), (1, 1), (1, −1), (2, 2), (2, −2) is not a function. 1 is paired with 1 and −1 and 2 is paired with 2 and −2.
(1, 1), (−1, 1), (2, 4), (−2, 4) is a function. Every value of *x* is paired with one and only one value of *y*.

DIRECTIONS: State whether or not each relation is a function. If the relation is not a function, tell why.

1. (1, 3), (2, 4), (3, 5), (4, 6)
2. (1, 1), (1, 2), (1, 3), (1, 4)
3. (1, 1), (2, 2), (3, 3), (4, 4)
4. (0, 0), (2, 4), (2, −4), (3, −9)
5. (3, 2), (3, −2), (4, 15), (4, −15)
6. (−2, −8), (−1, −1), (1, 1), (2, 8)
7. (2, 0), (2, 4), (2, −4), (4, 2)
8. (0, 0), (1, 1), (4, 2), (9, 3)
9. (−4, 2), (0, 3), (−4, 4), (0, 5)

CHALLENGE: (−2, 5), (−1, 0), (0, 1), and (2, 5) is a function. If the *x*-values and *y*-values of this function are switched, the relation that results is (5, −2), (0, −1), (1, 0), and (5, 2). This is not a function. Provide an example of a function that will still be a function if the *x*-values and *y*-values are switched.

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Teaching Notes 8.2: Finding the Domain of a Function

The rule, domain, and range are three components of functions. Students may assume that functions are defined for all these values and not realize that sometimes some values of functions are undefined.

1. Explain that all functions have a domain, range, and a rule that assigns to each member of the domain exactly one member of the range.
2. Review some common functions and their associated equations: linear, $f(x) = x$; squaring, $g(x) = x^2$; cubing, $h(x) = x^3$; and absolute value, $I(x) = |x|$. Sketch each graph using a t-table or calculator for your students. Using the equation and the graph, point out that for every value of x there is exactly one value for y. Also note that the domain of these functions is all real numbers.
3. Explain the square root function, $f(x) = \sqrt{x}$, and sketch the graph. Note that x must be greater than or equal to 0, otherwise $f(x)$ has no meaning.
4. Present $h(x) = \dfrac{1}{\sqrt{x}}$ and sketch the graph. Note that x must be greater than 0 because $h(x)$ is not defined when x is less than or equal to 0.
5. Review the information and the examples on the worksheet with your students. Emphasize the steps for finding restrictions on the variable. If necessary, review the steps for solving the equations in detail.

EXTRA HELP:

If the function is defined for all real numbers, the domain of the function is all real numbers.

ANSWER KEY:

(1) All real numbers
(2) All real numbers greater than or equal to 7
(3) All real numbers except -4
(4) All real numbers greater than -8
(5) All real numbers
(6) All real numbers
(7) All real numbers
(8) All real numbers except -2

(Challenge) Evie is incorrect. The domain of $f(x)$ is all real numbers greater than 1.

Name _____ Date _____

WORKSHEET 8.2: FINDING THE DOMAIN OF A FUNCTION

The domain of a function is the values of x for which the function is defined. The domains of most functions are all real numbers unless there are restrictions on the variable. Use the steps below to find restrictions on the variable:

1. If the variable is written within the square root symbol, write and solve the inequality that states the radicand must be greater than or equal to 0. This is the domain of the function.
2. If the variable is in the denominator, write and solve the inequality that states the denominator is not equal to 0. The domain is all real numbers except that value.
3. If the square root symbol is in the denominator, write and solve the inequality that states the radicand must be greater than 0. This is the domain of the function.

EXAMPLES

$f(x) = 3x^2$ There are no restrictions on x. The domain is all real numbers.

$g(x) = \dfrac{1}{x - 10}$ $x - 10 \neq 0 \rightarrow x \neq 10$ The domain is all real numbers except 10.

DIRECTIONS: Find the domain of each function.

1. $h(x) = x - 5$
2. $l(x) = \sqrt{x - 7}$
3. $g(x) = \dfrac{1}{x + 4}$
4. $j(x) = \dfrac{1}{2\sqrt{x + 8}}$
5. $f(x) = x^2 + 1$
6. $k(x) = |x - 3| + 1$
7. $h(x) = x^3 - 2$
8. $l(x) = \dfrac{1}{2x + 4}$

CHALLENGE: Evie said that the domain of $f(x) = \dfrac{1}{\sqrt{x - 1}}$ comprises all real numbers greater than or equal to 1. Do you agree? Explain your reasoning.

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Teaching Notes 8.3: Finding the Range of a Function

The members of the range of a function are the y-values of the function. Many students find function notation confusing and they often fail to make the correct substitutions when they try to find the range of a function.

1. Discuss the meaning of a function: A function consists of a domain, range, and a rule that assigns to each value of x exactly one value of y. Explain that the rule may be expressed as an equation or with arrow notation.
2. Offer some examples of functions, which are defined by an equation such as the absolute value function, $f(x) = |x|$; the square root function, $g(x) = \sqrt{x}$; and the reciprocal function, $F(x) = \frac{1}{x}$. Show your students how each can be written using arrow notation: $f : x \to |x|$, $g : x \to \sqrt{x}$, and $F : x \to \frac{1}{x}$. Explain how to interpret the notation. f pairs x with its absolute value; g pairs x with its square root; and F pairs x with its reciprocal.
3. Explain that the range of a function can be determined by substituting each member of the domain in the equation.
4. Review the information and example on the worksheet with your students. In the example, be sure that your students understand that "D" stands for domain and "R" stands for range.

EXTRA HELP:

When a member of the range is paired with two different members of the domain, write the member of the range only once.

ANSWER KEY:

(1) $\{-21, 28, 35\}$ **(2)** $\{3, 12\}$ **(3)** $\{0, 2, 3\}$ **(4)** $\{5, 4, 0\}$ **(5)** $\{4, 0\}$ **(6)** $\left\{-\frac{1}{2}, -1, 1\right\}$

(Challenge) Salvatore is correct. $f(-3) = 7$; $f(-2) = 4$; $f(-1) = 3$; $f(0) = 4$. The range is $\{7, 4, 3\}$. Each member of the range should be written only once.

Name _____ Date _____

WORKSHEET 8.3: FINDING THE RANGE OF A FUNCTION

You can find the range of a function if you are given members of the domain. Follow the steps below:

1. If the function is expressed in arrow notation, rewrite the function as an equation.
2. Substitute a member of the domain for x. Evaluate the expression to find the value of the function that is paired with the member of the domain. (Continue substituting until you have found the values of the function that are paired with every value of x.)
3. Rewrite the values of the function in braces. This represents the range of the function with the given domain.

EXAMPLE

Find the range. $D = \{-1, 1, 2\}$ $h : x \to x^2 + x$ or $h(x) = x^2 + x$

$h(-1) = (-1)^2 + (-1) = 0$ $h(1) = 1^2 + 1 = 2$ $h(2) = 2^2 + 2 = 6$ $R = \{0, 2, 6\}$

DIRECTIONS: Find the range of each function.

1. $h : x \to 7x; D = \{-3, 4, 5\}$
2. $l(x) = 3x^2; D = \{-1, 1, 2\}$
3. $j(x) = \sqrt{x + 4}; D = \{-4, 0, 5\}$
4. $F(x) \to |x - 5|; D = \{0, 1, 5\}$
5. $g(x) = x^2 - 3x; D = \{-1, 0, 3\}$
6. $f : x \to \dfrac{1}{x - 1}; D = \{-1, 0, 2\}$

CHALLENGE: Salvatore said that the range of $f(x) = x^2 + 2x + 4$ with the domain of $\{-3, -2, -1, 0\}$ was $\{7, 4, 3\}$. Michelle said he was wrong because there are four members of the domain and there must be four members of the range. Who is correct? Explain your answer.

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Teaching Notes 8.4: Using the Vertical Line Test

The vertical line test can be a useful tool to determine if the graph of a relation represents a function. Students often make mistakes applying the vertical line test to a graph, which leads to faulty interpretation of the results.

1. Explain that some relations are functions and other relations are not. If necessary, refer to 8.1: "Determining if a Relation Is a Function."
2. Explain that all relations and functions can be graphed in the coordinate plane. The vertical line test is a way to tell if a relation is a function by looking at its graph. Note that there are an infinite number of vertical lines that can be drawn in a coordinate plane. Some of these lines will intersect a graph of a relation and others will not. If there is at least one vertical line that intersects the graph at two or more points, then the relation is not a function.
3. Review the information and examples on the worksheet with your students. Emphasize that a vertical line must intersect the graph only once if the graph is the graph of a function.

EXTRA HELP:

Remember that every vertical line that can possibly be drawn must be considered in order to determine if a relation is a function.

ANSWER KEY:

(1) Function **(2)** Relation **(3)** Function **(4)** Relation

(Challenge) Alicia is correct. Every horizontal line is a function because every vertical line in the coordinate plane will intersect a horizontal line exactly once.

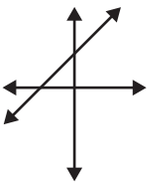
Name _____ Date _____

WORKSHEET 8.4: USING THE VERTICAL LINE TEST

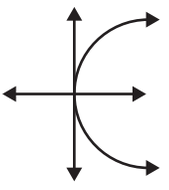
The vertical line test states that a relation is a function if no vertical line intersects the graph of the relation at two or more points. Follow the steps below to apply the vertical line test:

1. Study the graph of the relation. Try to find two or more points where a vertical line can be drawn to intersect the graph.
2. If you cannot find two or more points where a vertical line can be drawn to intersect the graph, the relation is a function.

EXAMPLES



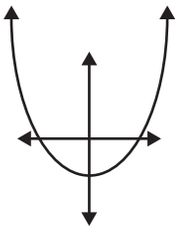
Function



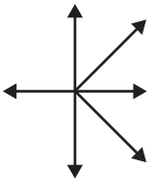
Relation

DIRECTIONS: State whether each relation is a function by applying the vertical line test.

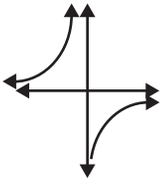
1.



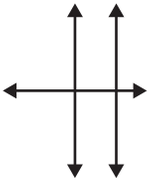
2.



3.



4.



CHALLENGE: Alicia concluded that every horizontal line is a function. Do you agree? Explain your reasoning.

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Teaching Notes 8.5: Describing Reflections of the Graph of a Function

Equations describe the reflections of the graph of a function. Students may confuse reflections in the y-axis with reflections in the x-axis because of where the negative sign is placed in the equation.

1. Show the graphs of the following basic functions to your students. Label each graph with the name of the function.

   Identity: $f(x) = x$ Squaring: $f(x) = x^2$ Cubing: $f(x) = x^3$

   Square root: $f(x) = \sqrt{x}$ Absolute value: $f(x) = |x|$ Reciprocal: $f(x) = \frac{1}{x}$

2. Explain that the graphs of functions may be reflected in the y-axis and in the x-axis. Note that when a graph of a function is reflected, the size and shape of the graph remains the same.

3. Consider the graph of the square root function $f(x) = \sqrt{x}$. Then sketch the graphs of $g(x) = -\sqrt{x}$ and $h(x) = \sqrt{-x}$. Note that the size and shape of all the graphs are the same. $g(x)$ is the reflection of $f(x)$ in the x-axis and $h(x)$ is the reflection of $f(x)$ in the y-axis.

4. Review the information and example on the worksheet with your students. For the example, emphasize that students must identify the basic function and then determine the axis in which the graph is reflected. Note that the domains of the functions are restricted so that the functions are defined.

EXTRA HELP:

The graph of the opposite of a function is a reflection of the graph of the function in the x-axis. The graph of the opposite of x is a reflection of the graph of the function in the y-axis.

ANSWER KEY:

The graph shows the following:

(1) Squaring function reflected in the x-axis
(2) Reciprocal function reflected in the x-axis
(3) Cubing function reflected in the y-axis
(4) Squaring function reflected in the y-axis
(5) Identity function reflected in the y-axis
(6) Cubing function reflected in the x-axis

(Challenge) The graphs look the same because when the graph of $y = x^2$ is reflected in the y-axis, the graphs coincide.

Name _____ Date _____

WORKSHEET 8.5: DESCRIBING REFLECTIONS OF THE GRAPH OF A FUNCTION

Equations can be used to show reflections of the graphs of a function. Follow the steps below:

1. Find the basic function the equation describes.
2. Compare the equation to $y = f(x)$.
   - The graph of $y = -f(x)$ is the reflection of $f(x)$ in the x-axis.
   - The graph of $y = f(-x)$ is the reflection of $f(x)$ in the y-axis.

EXAMPLE

Describe the graphs of $f(x) = -|x|$ and $g(x) = |-x|$.
The graph of $f(x)$ is the reflection of the absolute value function in the x-axis. The graph of $g(x)$ is the reflection of the absolute value function in the y-axis.

DIRECTIONS: Describe the graph of each function.

1. $f(x) = -x^2$
2. $g(x) = -\left(\frac{1}{x}\right)$
3. $F(x) = (-x)^3$
4. $h(x) = (-x)^2$
5. $j(x) = -x$
6. $l(x) = -(x)^3$

CHALLENGE: Explain why the graph of $y = x^2$ looks exactly like the graph of $y = (-x)^2$.

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Teaching Notes 8.6: Describing Vertical Shifts of the Graph of a Function

Using the equation that describes the graph of a function offers important information about vertical shifts. Mistakes often occur when students misinterpret the numbers given in the equation.

1. Show the graphs of the following basic functions to your students:

   Identity: $f(x) = x$ Squaring: $f(x) = x^2$ Cubing: $f(x) = x^3$

   Square root: $f(x) = \sqrt{x}$ Absolute value: $f(x) = |x|$ Reciprocal: $f(x) = \frac{1}{x}$

2. Explain that adding a positive number to a function moves the graph of a function up. Adding a negative number to a function moves the graph of a function down. For example, the graph of $f(x) = |x| + 3$ is the same size and shape as the graph of $f(x) = |x|$ but the graph is moved up 3 units. Similarly, the graph of $f(x) = x^2 - 2$ is the same size and shape as the graph of $f(x) = x^2$, except the graph is moved down 2 units. To illustrate these concepts, you may wish to sketch the graph of each function by plotting points.

3. Review the information and example on the worksheet with your students. Note that students must first identify the basic function and then find the value of d. Also note that the domains of the functions are restricted so that the functions are defined.

EXTRA HELP:

Vertical shifts do not change the shape of the graph of a function.

ANSWER KEY:

The graph shows the following:

(1) Reciprocal function shifted up 3 units.
(2) Cubing function shifted down 2 units.
(3) Squaring function shifted down 4 units.
(4) Absolute value function was not shifted.
(5) Square root function shifted up 2 units.
(6) Identity function shifted down 1 unit.

(Challenge) The domain of the function does not change. The range may change, depending on the function. For example, the range of $f(x) = x^3$ is the same as the range of $f(x) = x^3 + 1$. But the range of $g(x) = x^2$ is different from the range of $g(x) = x^2 + 1$.

Name _____ Date _____

WORKSHEET 8.6: DESCRIBING VERTICAL SHIFTS OF THE GRAPH OF A FUNCTION

Equations can be used to represent the vertical shifts of the graph of a function. Follow the steps below:

1. Find the basic function the equation describes.
2. Compare the equation to $y = f(x) + d$.
   - If $d > 0$, the graph is shifted up.
   - If $d < 0$, the graph is shifted down.
   - If $d = 0$, the graph is not shifted up or down.

EXAMPLE

Describe the graph of $f(x) = \sqrt{x} - 3$.
The graph is the same as the graph of the square root function but it is shifted down 3 units because $d = -3$.

DIRECTIONS: Describe the graph of each function.

1. $f(x) = \frac{1}{x} + 3$
2. $g(x) = x^3 - 2$
3. $h(x) = x^2 - 4$
4. $l(x) = |x|$
5. $j(x) = \sqrt{x} + 2$
6. $F(x) = x - 1$

CHALLENGE: Felipe said that when the graph of a function is moved vertically, the domain and range also change. Do you agree? Explain your answer.

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Teaching Notes 8.7: Describing Horizontal and Vertical Shifts of the Graph of a Function

Just as with vertical shifts, equations describe the horizontal shifts of the graph of a function. Students often confuse horizontal shifts with vertical shifts as the information of both can be obtained from the numbers in the equation.

1. Discuss the graphs of the following functions: identity, squaring, cubing, square root, absolute value, and reciprocal with your students. Also review the vertical shifts of these graphs. If necessary, refer to 8.6: "Describing Vertical Shifts of the Graph of a Function."
2. Explain that horizontal shifts are shifts to the right or left on the coordinate plane. Note that when a graph of a function is moved horizontally, the size and shape of the graph remains the same.
3. Sketch the following examples of graphs of functions by plotting points: $f(x) = x^2$ and $g(x) = (x - 3)^2$. Point out that the size and shape of both graphs are the same but $g(x) = (x - 3)^2$ is shifted 3 units to the right. Now sketch the graph of $h(x) = (x + 1)^2$ and note that it has the same size and shape as the graph of $f(x) = x^2$ but that the graph is shifted 1 unit to the left.
4. Review the information and example on the worksheet with your students. For the example, emphasize that students must identify the basic function and then find the values of c and d. Explain that $c = -4$ because $|x + 4|$ must be written as $|x - (-4)|$ in the formula. $d = -2$ so the graph is shifted down 2 units. Note that c tells the number of units the graph is shifted horizontally. d indicates the number of units the graph is moved vertically. Also note that the domains of the functions are restricted so that the functions are defined.

EXTRA HELP:

The number added to the value of x indicates the number of units of a horizontal shift.
The number added to the function indicates the number of units of a vertical shift.

ANSWER KEY:

The graph shows the following:

(1) Squaring function shifted 1 unit to the right and up 2 units.
(2) Cubing function shifted 2 units to the right and down 5 units.
(3) Reciprocal function shifted 3 units to the left.
(4) Reciprocal function shifted 3 units up.
(5) Absolute value function shifted 1 unit to the left and down 4 units.
(6) Square root function shifted 2 units to the left and down 6 units.

(Challenge) Mikal is correct. The positive 2 that is added to the function indicates a shift up 2 units.

Name _____ Date _____

WORKSHEET 8.7: DESCRIBING HORIZONTAL AND VERTICAL SHIFTS OF THE GRAPH OF A FUNCTION

Equations can be used to represent the horizontal shifts and vertical shifts of a function. Follow the steps below:

1. Find the basic function the equation describes.
2. Compare the equation to $y = f(x - c) + d$. c indicates a horizontal shift. d indicates a vertical shift.
3. Find the values of c and d.
   - If $c > 0$, the graph is shifted to the right.
   - If $c < 0$, the graph is shifted to the left.
   - If $c = 0$, the graph is not shifted right or left.
   - If $d > 0$, the graph is shifted up.
   - If $d < 0$, the graph is shifted down.
   - If $d = 0$, the graph is not shifted up or down.

EXAMPLE

Describe the graph of $f(x) = |x + 4| - 2$.

The graph is the same as the graph of the absolute value function but it is shifted 4 units to the left ($c = -4$) and 2 units down ($d = -2$).

DIRECTIONS: Describe the graph of each function.

1. $f(x) = (x - 1)^2 + 2$
2. $g(x) = (x - 2)^3 - 5$
3. $F(x) = \dfrac{1}{x + 3}$
4. $h(x) = \dfrac{1}{x} + 3$
5. $j(x) = |x + 1| - 4$
6. $l(x) = \sqrt{x + 2} - 6$

CHALLENGE: Mikal said that the graph of $f(x) = x + 2$ is the graph of the identity function shifted up 2 units. Annie said that the graph is the graph of the identity function shifted 2 units to the right. Who is correct? Explain your answer.

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Teaching Notes 8.8: Describing Dilations of the Graph of a Function

Unlike a reflection and vertical or horizontal shift whose graphs have the same size and shape, a dilation either stretches or compresses a graph. Students often have trouble recognizing the value that indicates the stretching or compressing of a graph.

1. Explain that a dilation can stretch a graph away from the x-axis or compress (or shrink) a graph toward the x-axis.
2. Show the graphs of the following basic functions to your students: identity, $f(x) = x$; squaring, $f(x) = x^2$; cubing, $f(x) = x^3$; square root, $f(x) = \sqrt{x}$; absolute value, $f(x) = |x|$; and reciprocal, $f(x) = \frac{1}{x}$.
3. Sketch the graphs of the following functions by drawing t-tables and plotting points: $f(x) = x^2$ and $g(x) = 3x^2$. Explain that the y-coordinate of each ordered pair in $g(x)$ is 3 times the value of the corresponding y-coordinate of $f(x)$. The graph of g is a stretching of the graph of f.
4. Sketch the graphs of the following functions by drawing t-tables and plotting points: $f(x) = x^2$ and $h(x) = \frac{1}{3}x^2$ or $\frac{x^2}{3}$. Explain that the y-coordinate of each ordered pair in $h(x)$ is $\frac{1}{3}$ the value of the corresponding y-coordinate of $f(x)$. The graph of h is a compression of the graph of f.
5. Review the information and example on the worksheet with your students. Emphasize that students must identify the basic function and whether it is a stretch or compression by looking at the value of a. Note that the domains of the functions are restricted so that the functions are defined.

EXTRA HELP:

Always rewrite the function with a numerical coefficient.

ANSWER KEY:

The graph shows the following:

(1) Reciprocal function stretched
(2) Absolute value function compressed
(3) Cubing function stretched
(4) Identity function compressed
(5) Squaring function stretched
(6) Square root function compressed

(Challenge) Louis is correct. A negative number represents a reflection in the x-axis and 2 shows a stretch of the graph.

Name _____ Date _____

WORKSHEET 8.8: DESCRIBING DILATIONS OF THE GRAPH OF A FUNCTION

Equations may be used to show a vertical stretch or a compression of the graph of a function. Follow the steps below:

1. Find the basic function the equation describes.
2. Compare the equation to $y = af(x)$.
   - If $a > 1$, the graph is stretched away from the x-axis.
   - If $0 < a < 1$, the graph is compressed (or shrunk) toward the x-axis.
   - If $a = 1$, there is no dilation.

EXAMPLE

Describe the graphs of $f(x) = 6|x|$ and $h(x) = \frac{|x|}{6}$.
The graph of $f(x)$ is the graph of the absolute value function. Because $a = 6$, which is greater than 1, it is stretched away from the x-axis. The graph of $h(x)$ is also the graph of the absolute value function. But because $a = \frac{1}{6}$, which is greater than 0 and less than 1, it is compressed toward the x-axis.

DIRECTIONS: Describe the graph of each function.

1. $g(x) = \frac{2}{x}$
2. $f(x) = \frac{2}{3}|x|$
3. $h(x) = 2x^3$
4. $j(x) = \frac{x}{2}$
5. $l(x) = 4x^2$
6. $F(x) = \frac{2}{5}\sqrt{x}$

CHALLENGE: Louis described the graph of $f(x) = -2|x|$ as the absolute value function reflected over the x-axis and stretched away from the x-axis. Is he correct? Explain your reasoning.

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Teaching Notes 8.9: Finding the Composite of Two Functions

To find the composite of two functions, students must substitute the range of one function into the domain of the other function. Many students find the notation of this process confusing.

1. Explain that a function is a rule that assigns to each member of the domain exactly one member of the range. Note that the domain is the input and the range is the output.
2. Provide your students with this example: $f(x) = 2x$. Explain that this function doubles the input. Note that the domain is x and the range is $2x$. Ask your students to find $f(-1)$, $f(3)$, and $f(10)$. (The answers are -2, 6, and 20.)
3. Explain that when the range of a function is substituted for the domain in another function, a composite function results. For example, suppose that the range of $g(x) = x + 3$ was placed in the f function above. Because the f function doubles the input, the output is $2(x + 3)$ or $2x + 6$. This can be expressed as $f(g(x)) = 2x + 6$. Note that the range of g is substituted in $f(x)$ and the function that results is the composite of f and g and is denoted by $f \circ g$.
4. Ask your students to explain the rule that applies to $g(x) = x + 3$, which is that this function adds 3 to the input. If the range of $f(x) = 2x$ was placed in the g function, 3 would be added to $2x$. This can be expressed as $g(f(x)) = 2x + 3$. Note that the range of f is substituted in $g(x)$ and the function that results is the composite of g and f and is denoted by $g \circ f$.
5. Review the information and example on the worksheet with your students. Emphasize that the range of the function that is substituted is enclosed in the outermost parentheses.

EXTRA HELP:

Correct substitution is very important. $f(g(x))$ may not equal $g(f(x))$.

ANSWER KEY:

(1) $3x^2 + 1$ **(2)** $16x^2 - 16x + 4$ **(3)** $12x - 5$ **(4)** $12x + 2$ **(5)** $4x^2$ **(6)** $-6x + 1$
(7) $9x^2 + 6x + 1$ **(8)** $-8x - 2$ **(9)** $-2x^2$ **(10)** $-6x - 2$ **(11)** $4x^2 - 2$ **(12)** $-8x + 4$

(Challenge) Bridget is correct. $j(k(x)) = k(j(x)) = x$.

Name _____ Date _____

WORKSHEET 8.9: FINDING THE COMPOSITE OF TWO FUNCTIONS

To find the composite of two functions, follow the steps below:

1. Find the two functions that will be combined.
2. Check the notation. The range of the function enclosed in parentheses will be substituted for x in the other function.
3. Simplify the output.

EXAMPLE

If $f(x) = 3x + 1$ and $h(x) = 4x - 2$, find $f(h(x))$.
The f and h functions will be combined. The range of $h(x)$, which is $4x - 2$, will be substituted for x in $f(x)$. $f(h(x)) = 3(4x - 2) + 1 = 12x - 5$

DIRECTIONS: Find the composites below by using these functions:
$f(x) = 3x + 1$, $g(x) = x^2$, $h(x) = 4x - 2$, and $I(x) = -2x$.

1. $f(g(x))$ **2.** $g(h(x))$ **3.** $f(h(x))$ **4.** $h(f(x))$

5. $g(I(x))$ **6.** $f(I(x))$ **7.** $g(f(x))$ **8.** $h(I(x))$

9. $I(g(x))$ **10.** $I(f(x))$ **11.** $h(g(x))$ **12.** $I(h(x))$

CHALLENGE: Bridget found two functions: $j(x) = x + 1$ and $k(x) = x - 1$. She said that $j(k(x)) = k(j(x))$. Do you agree? Explain your reasoning.

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Teaching Notes 8.10: Finding the Inverse of a Function

Finding the inverse of a function involves switching the values of x and y in the equation and solving for y. The idea of switching values often confuses students.

1. Ask your students to consider the function $f(x) = 2x$. Explain that the ordered pairs $(-1, -2)$, $(0, 0)$, $(1, 2)$, and $(3, 6)$ are solutions to $y = 2x$. The inverse of $f(x)$, denoted as $f^{-1}(x)$, is the function whose graph contains the points $(-2, -1)$, $(0, 0)$, $(2, 1)$, and $(6, 3)$. Note that the coordinates (x, y) of $f(x)$ are the (y, x) coordinates of $f^{-1}(x)$.
2. Review the information and examples on the worksheet with your students. Use the first example to verify that $(-2, -1)$, $(0, 0)$, $(2, 1)$, and $(6, 3)$ are the solutions to $f^{-1}(x) = \frac{x}{2}$. Note that the domains of the functions are restricted so that the functions are defined.

EXTRA HELP:

The inverse you found is correct if $f(f^{-1}(x)) = 1$.

ANSWER KEY:

(1) $f^{-1}(x) = \frac{1}{7}x$ **(2)** $g^{-1}(x) = -\frac{1}{4}x$ **(3)** $h^{-1}(x) = \sqrt[3]{x}$

(4) $l^{-1}(x) = \frac{x + 10}{3}$ **(5)** $F^{-1}(x) = \sqrt[3]{\frac{x + 9}{2}}$ **(6)** $k^{-1}(x) = x - 2$

(Challenge) Jeffrey is incorrect. $f^{-1}(x) = \pm\sqrt{x}$ is not a function because 4, for instance, is paired with 2 and -2.

Name _____ Date _____

WORKSHEET 8.10: FINDING THE INVERSE OF A FUNCTION

The inverse of a function $f(x)$ is denoted as $f^{-1}(x)$. To find the inverse of a function, follow the steps below:

1. Express the function as an equation, $y =$ an expression.
2. Replace y with x and x with y.
3. Solve the equation for y.
4. Rewrite the equation, replacing y with $f^{-1}(x)$.

EXAMPLES

Find the inverse of each function.

$f(x) = 2x$		$g(x) = x^3 + 9$
$y = 2x$		$y = x^3 + 9$
$x = 2y$	Switch x and y.	$x = y^3 + 9$
$\frac{2}{x} = y$	Solve for y.	$x - 9 = y^3$
$\frac{x}{2} = f^{-1}(x)$		$\sqrt[3]{x - 9} = y$
		$\sqrt[3]{x - 9} = g^{-1}(x)$

DIRECTIONS: Find the inverse of each function.

1. $f(x) = 7x$ **2.** $g(x) = -4x$ **3.** $h(x) = x^3$

4. $l(x) = 3x - 10$ **5.** $F(x) = 2x^3 - 9$ **6.** $k(x) = x + 2$

CHALLENGE: Jeffrey said that $f(x) = x^2$ has an inverse of $f^{-1}(x) = \pm\sqrt{x}$, which is also a function. Is he correct? Explain your answer.

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Teaching Notes 8.11: Evaluating the Greatest Integer Function

The greatest integer function is a special type of piecewise-defined function. Students often have trouble grasping the meaning of a piecewise-defined function and visualizing the graph. Once they master the concept, they must apply the definition of the greatest integer function to find the value of the function.

1. Explain that a piecewise-defined function is defined by two or more equations over a specific domain. Present an example by using the cost of mailing a letter. Let x represent the weight of the letter in ounces and y represent the cost. Therefore:
   - If $0 < x \leq 1$, $y = \$0.44$.
   - If $1 < x \leq 2$, $y = \$0.61$.
   - If $2 < x \leq 3$, $y = \$0.73$, and so on.
2. Sketch the graph for your students. Note the meaning of the inequality symbol and how the symbols are represented on the graph. An open circle represents the $<$ symbol. A closed circle represents the $\leq$ symbol.
3. Review the information and graph on the worksheet with your students. Point out that the smaller number is always to the left of a larger number on a number line. Review the examples and encourage your students to refer to the graph or a number line to evaluate the function. This will help them to identify the "piece" of the graph and use this to find the value of y.

EXTRA HELP:

Remember that integers are the set of counting numbers, their opposites, and zero.

ANSWER KEY:

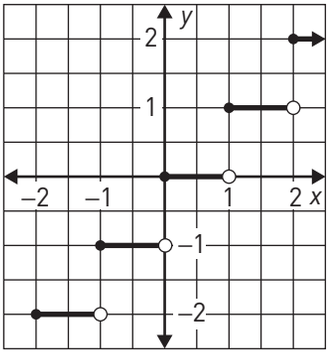
(1) 0 **(2)** 5 **(3)** -4 **(4)** -1 **(5)** 0 **(6)** -5

(Challenge) Terri is incorrect. She must rewrite the inequality as $4 \leq x < 5$.

Name _____ Date _____

WORKSHEET 8.11: EVALUATING THE GREATEST INTEGER FUNCTION

The greatest integer function is defined as the greatest (largest) integer that is less than or equal to x. It is denoted as $f(x) = [[x]]$. Its graph is shown below. Follow the steps below the graph to evaluate the greatest integer function.



$$[|x|] = \begin{cases} 2 \text{ if } 2 \le x < 3 \\ 1 \text{ if } 1 \le x < 2 \\ 0 \text{ if } 0 \le x < 1 \\ -1 \text{ if } -1 \le x < 0 \\ -2 \text{ if } -2 \le x < -1 \end{cases}$$

1. Visualize or draw a number line.
2. Find the value of x on the number line and determine which two integers x is between.
3. Use the number line to find the largest integer that is less than or equal to x.

EXAMPLES

$f(x) = [[x]]$. Find $f(1.5)$, $f(-1.5)$, and $f(2)$.
$f(1.5) = 1$. 1.5 is between 1 and 2. 1 is the largest integer that is less than 2.
$f(-1.5) = -2$. -1.5 is between -1 and -2. -2 is the largest integer that is less than -1.5.
$f(2) = 2$ because 2 is equal to 2.

DIRECTIONS: Find the values if $f(x) = [[x]]$.

1. $f(0)$
2. $f(5.2)$
3. $f(-3.5)$
4. $f(-1)$
5. $f(0.25)$
6. $f(-4.1)$

CHALLENGE: Terri tried to summarize all of the values of x so that $[[x]] = 4$. She said that $4 < x \le 5$. Is she correct? Explain your answer.

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Teaching Notes 8.12: Identifying Direct and Indirect Variation

Direct variation and indirect variation are easily confused. The quotient of the coordinates are constant in direct variation but the product of the coordinates are constant in indirect variation.

1. Explain that direct and indirect variation are functions that identify the way x- and y-coordinates are related.
2. Offer this example of direct variation: If one movie ticket costs \$8, then two tickets cost \$16, three tickets cost \$24, and so on. Each pair of numbers can be represented as an ordered pair, (x, y). (1, 8), (2, 16), (3, 24) The quotient $\frac{y}{x}$ of each ordered pair is 8.
3. Offer this example of indirect variation: Suppose a DJ is hired to play at a community picnic (for entertainment only and not to make a profit for the picnic organizers). If the DJ charges \$500 and 50 people attend the picnic, the cost is \$10 per person. If 100 people attend, the cost is \$5 per person. If 25 people attend, the cost is \$20 per person. Each pair of numbers can be represented as an ordered pair. (50, 10), (100, 5), (25, 20) The product xy of each ordered pair is 500.
4. Discuss the information, formulas, and example on the worksheet with your students. Be sure to relate both formulas to the example.

EXTRA HELP:

Consider every ordered pair. For a direct variation, the quotient of every ordered pair is the same. For an indirect variation, the product of every ordered pair is the same.

ANSWER KEY:

(1) Direct **(2)** Neither **(3)** Indirect **(4)** Direct **(5)** Direct **(6)** Indirect

(Challenge) Brittany is incorrect. The quotient $\frac{y}{x}$ of the first two ordered pairs is $-\frac{1}{2}$ but the quotient $\frac{y}{x}$ of the last ordered pair is $\frac{6}{-3}$ or -2.

Name _____ Date _____

WORKSHEET 8.12: IDENTIFYING DIRECT AND INDIRECT VARIATION

Direct and indirect variations are functions that show how x-coordinates and y-coordinates are related. Use the formulas below to find if a relation is direct or indirect. (k is a constant.)

Direct variation: $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ *Indirect variation:* $x_1 y_1 = x_2 y_2$

1. To determine if a relation is a direct variation, find the quotient $\frac{y}{x}$ of each ordered pair. If the quotients of the ordered pairs are the same, the variation is direct.
2. To determine if a relation is an indirect variation, find the product xy of each ordered pair. If the products of the ordered pairs are the same, the variation is indirect.

EXAMPLE

State if the coordinates describe a direction variation, an indirect variation, or neither. (2, 5), (−1, −10), (−4, −2.5)

Because $\frac{5}{2} \neq \frac{-10}{-1}$, the coordinates *do not* describe a direct variation.

Because $2 \cdot 5 = -1(-10) = -4(-2.5) = 10$, the coordinates describe an indirect variation.

DIRECTIONS: Determine if the coordinates describe a direct variation, an indirect variation, or neither.

1. (1, 2), (2, 4), (3, 6)
2. (2, 8), (1, 16), (−4, 4)
3. (6, 5), (5, 6), (−3, −10)
4. (2, 2), (4, 4), (8, 8)
5. $\left(\frac{1}{2}, 2\right)$, (2, 8), (−1, −4)
6. (4, 3), $\left(\frac{3}{4}, 16\right)$, (6, 2)

CHALLENGE: Brittany said that (−2, 1), (−4, 2), and (−3, 6) is a direct variation because the quotient of each ordered pair is $-\frac{1}{2}$. Is she correct? Explain your answer.

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Teaching Notes 8.13: Describing the Graph of the Quadratic Function

The quadratic equation provides information about the vertex and axis of symmetry of the graph of the function. Some students have trouble organizing and relating all of the information needed to describe the graph.

1. Review the squaring function, $f(x) = x^2$, with your students. Explain that it is a parabola that opens upward. The opposite of this function, $g(x) = -x^2$, is a reflection of the parabola in the x-axis. It opens downward. Note that the graphs of all squaring functions are parabolas.
2. Explain that $f(x) = ax^2 + bx + c$ is the general form of the squaring function. If $a > 0$, the graph is a parabola that opens upward. If $a < 0$, the graph opens downward.
3. Explain that all parabolas have a vertex, which is the lowest point of the graph if it opens upward or the highest point if it opens downward. By looking at the equation, students should be able to determine the x-coordinate of the vertex, which is $-\frac{b}{2a}$. Students can then substitute this value into the equation and solve for the y-coordinate of the vertex.
4. Explain that every parabola has an axis of symmetry, a line where a parabola can be folded so that each side of the graph will coincide. The equation of the axis of symmetry is $x = -\frac{b}{2a}$.
5. Review the information and example on the worksheet with your students. You may wish to sketch the graph by plotting points to verify the vertex and axis of symmetry.

EXTRA HELP:

The line of symmetry will always intersect the vertex of the parabola.

ANSWER KEY:

(1) Minimum point $\left(\frac{1}{2}, \frac{7}{4}\right)$; $x = \frac{1}{2}$

(2) Maximum point $\left(\frac{-3}{2}, \frac{13}{4}\right)$; $x = \frac{-3}{2}$

(3) Minimum point $(0, -1)$; $x = 0$

(4) Maximum point $\left(\frac{3}{2}, -\frac{9}{2}\right)$; $x = \frac{3}{2}$

(Challenge) If $a = 0$ was substituted into the quadratic equation, $y = ax^2 + bx + c$, the resulting equation would be $y = bx + c$, which is a linear equation.

Name _____ Date _____

WORKSHEET 8.13: DESCRIBING THE GRAPH OF THE QUADRATIC FUNCTION

The graph of every quadratic function is a parabola. The equation $y = ax^2 + bx + c, a \neq 0$, can be used to describe the vertex and axis of symmetry of the graph. Follow the steps below:

1. Identify a and b.
2. Determine the sign of a. If a is positive, the graph has a minimum point. If a is negative, the graph has a maximum point.
3. Find the x-coordinate of the vertex. Use the formula $x = -\frac{b}{2a}$.
4. Find the y-coordinate of the vertex. Substitute $x = -\frac{b}{2a}$ in the equation and solve for y.
5. Use the x- and y-coordinates to write the vertex.
6. Find the axis of symmetry. The line $x = -\frac{b}{2a}$ is the axis of symmetry.

EXAMPLE

Describe the graph of $y = x^2 + 4x - 3$.
$a = 1$ and $b = 4$.
Because $a > 0$, the graph has a minimum point.
The x-coordinate of the vertex is $x = -\frac{4}{2 \cdot 1} = -2$.
The y-coordinate of the vertex is $y = 1(-2)^2 + 4(-2) - 3 = -7$.
The vertex is $(-2, -7)$.
The axis of symmetry is the line $x = -2$.

DIRECTIONS: Describe the graph of each equation.

1. $y = x^2 - x + 2$
2. $y = -x^2 - 3x + 1$
3. $y = x^2 - 1$
4. $y = -2x^2 + 6x - 9$

CHALLENGE: Explain why the coefficient of a quadratic equation cannot equal 0.

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Teaching Notes 8.14: Using Rational Numbers as Exponents

Rational numbers have a special meaning when they are used as exponents. The denominator indicates the root of the number and the numerator indicates the number of times the root is used as a factor. Students often confuse the meaning of the denominator with the meaning of the numerator. They may also fail to realize that negative exponents represent reciprocals.

1. Review the meaning of the roots of numbers by providing the following examples. The square root of 36 is 6, which can be written as $\sqrt{36} = 6$. Note that finding the square root of a number is the opposite of squaring the number, $6^2 = 36$. Also review that the cube root of 8 is 2, which can be written as $\sqrt[3]{8} = 2$. Finding the cube root of a number is the opposite of cubing the number, $2^3 = 8$.
2. Explain to your students that the roots of numbers can be expressed with exponents. For example, $36^{\frac{1}{2}}$ can be expressed as $\sqrt{36}$. Another example is $8^{\frac{1}{3}} = \sqrt[3]{8}$. The denominator of the exponent indicates the root of the number. If there is no number in the indentation of the radical symbol, the square root is implied. Emphasize that the numerator indicates the number of times the root is used as a factor.
3. Review the information and examples on the worksheet with your students. Explain that exponents may be either positive or negative. If the exponent is negative, students must write the reciprocal of the expression and use the positive exponent. The meanings of the numerator and denominator do not change.

EXTRA HELP:

Remember to write a reciprocal if the exponent is negative.

ANSWER KEY:

(1) 343 **(2)** 2 **(3)** $\frac{1}{25}$ **(4)** 1,000 **(5)** $\frac{1}{8}$ **(6)** $\frac{1}{100}$ **(7)** 64 **(8)** $\frac{1}{36}$

(Challenge) Both methods are correct. In this case, it is easier to simplify the exponent first.

Name _____ Date _____

WORKSHEET 8.14: USING RATIONAL NUMBERS AS EXPONENTS

Follow the rules below when simplifying expressions that have rational numbers as exponents:

If the rational exponent is positive, $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$:

- The denominator of the exponent indicates the root of the number. Write the base using the denominator as the index and evaluate the expression.
- Multiply the expression the number of times the base is used as a factor. The numerator indicates the number of times the base is used as a factor.

If the rational exponent is negative, $x^{-\frac{m}{n}} = \frac{1}{x^{\frac{m}{n}}} = \frac{1}{(\sqrt[n]{x})^m}$:

- Place "1" over the expression, which makes the exponent positive.
- Then follow the steps above.

EXAMPLES

$27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$ $\qquad$ $27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{9}$

DIRECTIONS: Simplify each expression.

1. $49^{\frac{3}{2}}$ **2.** $16^{\frac{1}{4}}$ **3.** $125^{-\frac{2}{3}}$ **4.** $100^{\frac{3}{2}}$

5. $64^{-\frac{1}{2}}$ **6.** $1000^{-\frac{2}{3}}$ **7.** $16^{\frac{3}{2}}$ **8.** $216^{\frac{-2}{3}}$

CHALLENGE: Victor simplified $16^{\frac{4}{2}}$ as $(\sqrt{16})^4$ or 4^4, which equals 256. Ben rewrote the exponent $\frac{4}{2}$ as 2 and then said that $16^2 = 256$. Are both methods correct? Explain your answer.

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Teaching Notes 8.15: Using Irrational Numbers as Exponents

The same properties of exponents that apply to rational exponents also apply to irrational exponents. A common mistake students make with these problems is to express one number in terms of the base of another number. Other common mistakes include applying the properties of exponents incorrectly.

1. Explain that rational numbers can be expressed as fractions; irrational numbers cannot be expressed as fractions. Some examples of rational numbers include 3, 0.24, -21, and $7.\overline{253}$. Examples of irrational numbers include $\sqrt{2}$, $\sqrt{5}$, and π.
2. Provide this example of a base with a rational exponent: $2^3 = 2 \cdot 2 \cdot 2 = 8$. Now provide this example of a base with an irrational exponent: 2^π equals 2 multiplied by itself π times. Because π is an irrational number (it is a nonterminating, nonrepeating decimal), 2^π has no exact value.
3. Review the information and examples on the worksheet with your students. Explain that the properties of exponents apply to both rational and irrational exponents. You may find it helpful to review 6.3: "Using the Properties of Exponents That Apply to Multiplication and Division." Emphasize that the base of the numbers must be the same before students can apply the properties of exponents. If the bases are different, students must rewrite one or both of the numbers so that the bases are the same. Be sure to discuss the examples fully. In particular, note that in the second example, 49 was rewritten as 7^2, and in the third example, both 8 and 16 must be rewritten as a power of 2. In these examples, the problems had to be rewritten so that the bases are the same.

EXTRA HELP:

Be sure the bases are the same before you apply the properties of exponents.

ANSWER KEY:

(1) $4^{2\sqrt{2}}$ **(2)** $3^2 = 9$ **(3)** $5^{\sqrt{2}-2}$ **(4)** $2^{6+2\sqrt{2}}$ or $4^{3+\sqrt{2}}$ **(5)** $6^{2\sqrt{7}-2}$ **(6)** $5^{4-3\sqrt{2}}$

(Challenge) Leah is incorrect. She tried to multiply the expressions by adding the exponents but the bases are different. She should have rewritten $9^{\sqrt{2}}$ as $3^{2\sqrt{2}}$. The correct answer is $3^{3\sqrt{2}}$.

Name _____ Date _____

WORKSHEET 8.15: USING IRRATIONAL NUMBERS AS EXPONENTS

The properties of exponents that apply to expressions with rational exponents also apply to expressions with irrational exponents. Follow the guidelines below to simplify expressions with irrational exponents:

1. If the bases are the same:
   - To multiply two expressions, add the exponents.
   - To raise a power to a power, multiply the exponents.
   - To divide two expressions, subtract the exponent in the denominator from the exponent in the numerator.
2. If the bases are different, express each number in terms of the same base. Then follow the guidelines for the bases being the same.

EXAMPLES

$8^{\sqrt{2}} \cdot 8^{\sqrt{3}} = 8^{\sqrt{2}+\sqrt{3}}$

$49^{\sqrt{2}} \cdot 7^2 = (7^2)^{\sqrt{2}} \cdot 7^2 = 7^{2\sqrt{2}} \cdot 7^2 = 7^{2\sqrt{2}+2}$

$\dfrac{8^{\sqrt{2}}}{16^{\sqrt{3}}} = \dfrac{(2^3)^{\sqrt{2}}}{(2^4)^{\sqrt{3}}} = 2^{3\sqrt{2}-4\sqrt{3}}$

DIRECTIONS: Simplify.

1. $4^{\sqrt{2}} \cdot 4^{\sqrt{2}}$
2. $(3^{\sqrt{2}})^{\sqrt{2}}$
3. $\dfrac{5^{\sqrt{2}}}{5^2}$
4. $8^2 \cdot 4^{\sqrt{2}}$
5. $\dfrac{36^{\sqrt{7}}}{6^2}$
6. $\dfrac{25^2}{125^{\sqrt{2}}}$

CHALLENGE: Leah simplified $3^{\sqrt{2}} \cdot 9^{\sqrt{2}}$ as $9^{2\sqrt{2}}$. Is she correct? Explain your answer.

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Teaching Notes 8.16: Solving Exponential Equations

Solving exponential equations requires students to express each side of the equation as a power of the same base. Sometimes students forget to write both sides of the equation as two expressions that have the same base or they write an incorrect equation.

1. Explain that an exponential equation is an equation that has a variable in the exponent. For example, $9^2 = 3^x$ and $2^{3x} = 2^{x-1}$. Note that the variable in the exponent may be on only one side or both sides of the equation.
2. Explain that if the two bases are the same, then the equation must be true if the exponents have the same value. Students can then write an equation stating that the two exponents are equal and then solve the equation.
3. Explain that if the bases are different, students must rewrite the equation so that the bases are the same. Depending on the abilities of your students, you may wish to review some of the more common powers, for example, $2^2 = 4$, $2^3 = 8$, $4^2 = 16$, $5^3 = 125$, and so on.
4. Review the information and examples on the worksheet with your students. Note that in the first example, students do not have to rewrite the equation because the common base is 2. In the second example, 9^2 is expressed as $(3^2)^2$ or 3^4. The equation in the second example could also have been rewritten as $9^2 = (9^{\frac{1}{2}})^x$. Remind your students that when they find powers of powers, they should multiply the exponents.

EXTRA HELP:

This method may be used only when one base is raised to a power on each side of the equation.

ANSWER KEY:

(1) $x = -1$ **(2)** $x = 3$ **(3)** $x = 1$ **(4)** $x = \frac{3}{2}$ **(5)** $x = 0$ **(6)** $x = 5$

(Challenge) Both are correct. The bases in the equation must be the same. In this case, 9 or 3 may be used as the base.

Name _____ Date _____

WORKSHEET 8.16: SOLVING EXPONENTIAL EQUATIONS

An exponential equation is an equation that includes a variable in the exponent. Follow the steps below to solve an exponential equation:

1. Rewrite each side of the equation so that the bases are the same, if necessary.
2. Write an equation stating that the two exponents are equal.
3. Check your answer by substituting the values in the original equation.

EXAMPLES

$2^{3x} = 2^{x-1}$

Solve for x:

$3x = x - 1$

$2x = -1$

$x = -\frac{1}{2}$

Check: $2^{-\frac{3}{2}} = 2^{-\frac{1}{2}-1}$

$9^2 = 3^x$

Rewrite the equation: $(3^2)^2 = 3^x \rightarrow 3^4 = 3^x$

Solve for x: $x = 4$

Check: $9^2 = 3^4$ or $81 = 81$

DIRECTIONS: Solve.

1. $3^{4x} = 3^{2x-2}$
2. $5^{x-1} = 25$
3. $49 = 7^{3x-1}$
4. $4 = 16^{x-1}$
5. $2^{x+1} = \sqrt{4}$
6. $8 = 2^{x-2}$

CHALLENGE: Myles rewrote $9^{4x} = 81$ as $9^{4x} = 9^2$. Claire rewrote $9^{4x} = 81$ as $(3^2)^{4x} = 3^4$ or $3^{8x} = 3^4$. In each case, the answer is $x = \frac{1}{2}$. Who is correct? Explain your answer.

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Teaching Notes 8.17: Using the Compound Interest Formula

Because of the complexity of the compound interest formula, students often have difficulty understanding how to solve compound interest problems. Incorrect substitution of values for variables and computational mistakes are particularly common.

1. Explain to your students that the compound interest formula is used by a bank or investment company to determine payment of interest on the principal and any unpaid interest accumulated in an account.
2. Review the formula and information on the worksheet with your students. Define all the variables, making sure that your students understand the terms of the formula. Make special note of the variables n and t. Both variables must be expressed in terms of the number of years. For example, if interest was compounded semiannually, n would equal 2. If money was deposited for 6 months, t would equal 0.5.
3. Offer the following example: An amount of $1,000 is deposited in a bank at an annual interest rate of 2.4 percent, compounded quarterly. No money is withdrawn from the account. What is the principal after five years? Note that $r = 0.024$; $n = 4$ because interest is compounded four times a year and $t = 5$ because the money is invested for five years. Show students how to determine the value of each variable, substitute the values in the equation, and solve for A. ($P = 1000, r = 0.024, n = 4, t = 5; A = \$1{,}127.09$)

EXTRA HELP:

Be sure to substitute the correct values in the formula.

ANSWER KEY:

(1) $797.91 **(2)** $1,136.35 **(3)** $2,615.20 **(4)** $3,904.82

(Challenge) Bank A, because she would earn $20 in interest by the end of the year. In Bank B, she would earn about $17.62 by the end of the year.

Name _____ Date _____

WORKSHEET 8.17: USING THE COMPOUND INTEREST FORMULA

Compound interest is interest paid on the principal and accumulated unpaid interest in a bank account or other investment. The following formula is used to calculate compound interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

A = the amount of money accumulated after n years
P = the principal amount (initial amount of money deposited)
r = the annual rate of interest (expressed as a decimal)
n = the number of times interest is compounded per year
t = the number of years the amount is deposited

DIRECTIONS: Solve each problem using the compound interest formula. Round your answer to the nearest hundredth. (For each problem, assume that the person does not withdraw any of the principal or accumulated interest.)

1. Sarah placed $750 in an account in the bank. The bank pays an interest rate of 3.1%, compounded monthly. How much money will she have after two years?

2. Tom put $900 in an account at an interest rate of 4.7%, compounded every four months. How much money will he have after five years?

3. Jamal invested $2,000 at an interest rate of 2.7%, compounded semiannually. How much money will he have after ten years?

4. Diana invested $1,500 in long-term bonds at an interest rate of 4.9%, compounded yearly. How much money will she have after twenty years?

CHALLENGE: Danielle is considering putting $1,000 into a savings account for one year. Bank A offers an interest rate of 2%, compounded annually. Bank B offers an interest rate of 1.75%, compounded quarterly. Which bank offers the better deal? Explain your answer.

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Teaching Notes 8.18: Solving Radical Equations

To solve radical equations, which are equations that contain a variable in the radicand, students must follow the same steps as they would to solve multistep equations but with one additional step. They must square both sides of an equation to eliminate the radical symbol. A common mistake is to ignore the radical symbol when trying to solve the equation. Another common mistake is not to substitute their answer in the original equation to check it.

1. Explain that a radical equation is an equation that has a variable in the radicand.
2. Provide the following example: $\sqrt{x} = 5$. Ask your students what they think x equals. Most will quickly tell you the answer is 25. Ask how they arrived at this answer. They might say that they know $5^2 = 25$. This concept will help them make the connection between squaring a number and its square root.
3. Review the information and examples on the worksheet with your students. Emphasize that squaring both sides of an equation may produce a solution that is not a solution to the original equation. Note that in the second example, the radical is isolated and it is equal to a negative number. Therefore, there is no real solution. Explain that if students were to continue and solve for x, and then substitute their answer into the original equation, their answer would not check.

EXTRA HELP:

If you square a radical, the product is equal to the radicand.

ANSWER KEY:

(1) $x = 9$ **(2)** $x = 16$ **(3)** No real solution **(4)** $x = 618$
(5) $x = 33$ **(6)** $x = 27$ **(7)** $x = 32$ **(8)** $x = 33$

(Challenge) Jimmy is incorrect. He has to look at the entire equation. If he divides both sides by -1 to isolate the radical, $\sqrt{x+2} = 4$, the solution is $x = 14$.

Name _____ Date _____

WORKSHEET 8.18: SOLVING RADICAL EQUATIONS

Follow the steps below to solve radical equations:

1. Isolate the radical.
2. Square both sides of the equation. This will eliminate the radical symbol.
3. Solve.
4. Check the solution by substituting the value in the original equation.

EXAMPLES

Solve.

$\sqrt{x} + 3 = 5$ $\qquad$ $\sqrt{2x} + 4 = -2$

$\sqrt{x} = 2$ $\qquad$ $\sqrt{2x} = -6$ $\qquad$ The square root of any number must be ≥ 0.

$(\sqrt{x})^2 = 2^2$ $\qquad$ There is no real solution.

$x = 4$

Check: $\sqrt{4} + 3 = 5$

$2 + 3 = 5$

DIRECTIONS: Solve each equation.

1. $\sqrt{x} + 5 = 8$
2. $\sqrt{x} - 4 = 0$
3. $2\sqrt{x} + 10 = 6$
4. $\sqrt{x + 7} - 4 = 21$
5. $4 - \sqrt{x + 3} = -2$
6. $\sqrt{2x - 5} = 7$
7. $10 + \sqrt{3x + 4} = 20$
8. $15 - \sqrt{3x + 1} = 5$

CHALLENGE: Jimmy looked at the equation $-\sqrt{x + 2} = -4$. He said that there is no real solution because the radical equals -4 and square roots are always positive. Do you agree? Explain your reasoning.

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Teaching Notes 8.19: Writing Logarithmic Equations as Exponential Equations

When students are given a logarithmic equation, they can express it as an exponential equation. They must understand where to substitute the *y*-value, the base, and the *x*-value.

1. Explain that an exponential function has a numerical base and an exponent that is a variable. $f(x) = b^x$ is the general form.
2. Explain that the logarithmic function, $g(x) = \log_b x$, is the inverse of the exponential function.
3. Explain that the *y*-values (the range) of any function are *x*-values (the domain) of its inverse. Depending on the abilities of your students, you may wish to illustrate this concept by using two functions your students are familiar with, for example, the squaring function, $f(x) = x^2$, and its inverse, the square root function, $g(x) = \sqrt{x}$. $f(2) = 4$, therefore $g(4) = 2$; $f(6) = 36$, therefore $g(36) = 6$.
4. Expand this concept to the exponential function, $f(x) = b^x$, and its inverse, $g(x) = \log_b x$. For example, if the base is 3, $f(2) = 3^2$ or 9, therefore $g(9) = \log_3 9 = 2$ because $3^2 = 9$. If the base is 3, $f(4) = 3^4$ or 81, therefore $g(81) = \log_3 81 = 4$ because $3^4 = 81$. Be sure to point out to your students that the value of the logarithmic function is the same as the exponent in the exponential function.
5. Review the information and examples on the worksheet with your students. Make sure that they understand all the steps of the examples.

EXTRA HELP:

The subscript in the logarithmic function is the base in the exponential function.

ANSWER KEY:

(1) $2^3 = 8$ **(2)** $25^{\frac{1}{2}} = 5$ **(3)** $7^2 = 49$ **(4)** $16^{\frac{3}{2}} = 64$ **(5)** $8^{-\frac{2}{3}} = \frac{1}{4}$

(Challenge) $\log_{\sqrt{5}} 5 = 2$

Name _____ Date _____

WORKSHEET 8.19: WRITING LOGARITHMIC EQUATIONS AS EXPONENTIAL EQUATIONS

A logarithmic equation of the form $y = \log_b x$ can be rewritten as an exponential equation of the form $x = b^y$. Follow the steps below:

1. Use the logarithmic equation to identify y, b, and x. Remember that b is always the base.
2. Substitute these values in the exponential equation, $x = b^y$.
3. Check your work.

EXAMPLES

$2 = \log_8 64$

$y = 2, b = 8, x = 64$

$x = b^y \rightarrow 64 = 8^2$

$8 \cdot 8 = 64$

$\log_{36} 216 = \frac{3}{2}$

$y = \frac{3}{2}, b = 36, x = 216$

$x = b^y \rightarrow 216 = 36^{\frac{3}{2}}$

$36^{\frac{1}{2}} = 6, 6^3 = 216$

DIRECTIONS: Write an exponential equation for each.

1. $3 = \log_2 8$
2. $\frac{1}{2} = \log_{25} 5$
3. $\log_7 49 = 2$
4. $\frac{3}{2} = \log_{16} 64$
5. $\log_8 \frac{1}{4} = -\frac{2}{3}$

CHALLENGE: Louis copied a logarithmic equation from the board incorrectly as $\log_{\sqrt{5}} = 5$. The correct exponential equation is $(\sqrt{5})^2 = 5$. What is the correct logarithmic equation?

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Teaching Notes 8.20: Solving Logarithmic Equations

To solve a logarithmic equation, students must write the equation in exponential form. A common mistake is failing to remember that the logarithm of a number is an exponent.

1. Explain that logarithmic equations can be rewritten as exponential equations. Depending on the abilities of your students, you may find it helpful to review 8.19: "Writing Logarithmic Equations as Exponential Equations." Emphasize that the logarithm of a number is the exponent. For example, $\log_6 x = 2$ can be rewritten as $6^2 = x$.
2. Explain that once the exponential equation is written, the equation can be solved.
3. Review the information and examples on the worksheet with your students. Note that there are three types of equations: one in which a number is the base and a number is the exponent (as in the first example); another in which the base is a variable (as in the second example); and a third in which an exponent is a variable (as in the third example). The first equation is solved by evaluating the expression. To solve the second equation, students must ask themselves what number raised to the third power is 216. They may use guess and check to solve the equation or find the cube root of 216. To solve the third equation, students must rewrite the equation so that the bases are the same. They must then solve for x.

EXTRA HELP:

Double-check that the equation is written correctly before you try to solve it.

ANSWER KEY:

(1) $x = \frac{1}{2}$ **(2)** $x = 3$ **(3)** $x = \frac{1}{3}$ **(4)** $x = 7$ **(5)** $x = 5$

(6) $x = -2$ **(7)** $x = 2$ **(8)** $x = \frac{1}{4}$ **(9)** $x = 3$ **(10)** $x = 6$

(Challenge) Elena is incorrect. She rewrote the equation incorrectly. It should be $100^x = 10$, therefore $x = \frac{1}{2}$.

Name _____ Date _____

WORKSHEET 8.20: SOLVING LOGARITHMIC EQUATIONS

A logarithmic equation is an equation that contains a logarithmic expression. Solve logarithmic equations by following the steps below:

1. Rewrite the equation in exponential form.
2. Solve the exponential equation.

EXAMPLES

$\log_2 x = 5$
$2^5 = x$
$32 = x$

$\log_x 216 = 3$
$x^3 = 216$
$x = 216^{\frac{1}{3}}$ or $x = 6$

$\log_3 81 = x$
$3^x = 81;\ 81 = 3^4$
$3^x = 3^4$ or $x = 4$

DIRECTIONS: Solve.

1. $\log_4 2 = x$

2. $\log_5 125 = x$

3. $\log_8 2 = x$

4. $\log_x \frac{1}{7} = -1$

5. $\log_{25} x = \frac{1}{2}$

6. $\log_6 \frac{1}{36} = x$

7. $\log_8 64 = x$

8. $\log_{16} x = -\frac{1}{2}$

9. $\log_x 243 = 5$

10. $\log_2 64 = x$

CHALLENGE: Elena solved $\log_{100} 10 = x$ and found that $x = 2$ because $10^2 = 100$. Is she correct? Explain your reasoning.

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Teaching Notes 8.21: Using the Properties of Logarithms

The properties of logarithms can be used to express the sum, difference, or product of a number (or numbers) and a logarithm as a simple logarithm. Many students encounter trouble when they apply the properties of logarithms, particularly when applying them in reverse.

1. Explain the three properties of logarithms to your students and provide examples. Note that a is the base of the logarithmic function. $a > 0$, $a \neq 1$.

First property:	$\log_a MN = \log_a M + \log_a N$
	$\log_2 16 = \log_2 8 + \log_2 2$
Check:	$4 = 3 + 1$
Second property:	$\log_a \frac{M}{N} = \log_a M - \log_a N$
	$\log_2 \frac{16}{2} = \log_2 16 - \log_2 2$
Check:	$3 = 4 - 1$
Third property:	$\log_a M^k = k \log_a M$
	$\log_2 8^5 = 5 \log_2 8$
	$\log_2 8^5 = 5 \cdot 3$
Check:	$8^5 = 2^{15}$

2. Explain that logarithms may be condensed by applying the properties of logarithms in reverse.
3. Review the information and examples on the worksheet with your students. Note that the base is a, which can represent any base. Thus, students will never be able to find a specific number.

EXTRA HELP:

Always apply the third property before applying the first or second property.

ANSWER KEY:

(1) $\log_a 5$ **(2)** $\log_a 36$ **(3)** $\log_a 4$ **(4)** $\log_a 12$

(Challenge) Terri's method is wrong. She should have simplified $\frac{1}{2} \log_a 64$ as $\log_a 64^{\frac{1}{2}}$ or $\log_a 8$ first. Then she should have rewritten $\log_a 8 + \log_2 2$ as $\log_a 16$.

Name _____ Date _____

WORKSHEET 8.21: USING THE PROPERTIES OF LOGARITHMS

The properties of logarithms may be used to express a sum, difference, or product of a logarithm as a single logarithm. Follow the steps below:

1. Identify which of the following properties apply to the problem:
   - First property: $\log_a MN = \log_a M + \log_a N$
   - Second property: $\log_a \frac{M}{N} = \log_a M - \log_a N$
   - Third property: $\log_a M^k = k \log_a M$
2. Apply the third property before using the first or second property.
3. Simplify, if possible.

EXAMPLES

Write each expression as a single logarithm.

Example 1: Only the first property applies.

$$\log_a 5 + \log_a 4 = \log_2 20$$

Example 2: Apply the third property, then apply the second property.

$$3 \log_a 4 - \log_a 2 = \log_a 4^3 - \log_a 2 = \log_a 64 - \log_a 2 = \log_a \frac{64}{2} = \log_a 32$$

DIRECTIONS: Write the expressions as a single logarithm.

1. $\log_a 10 - \log_a 2$
2. $2 \log_a 3 + \log_a 4$
3. $-\log_a \frac{1}{4}$
4. $\frac{1}{2} \log_a 16 + \log_a 3$

CHALLENGE: Terri rewrote $\frac{1}{2} \log_a 64 + \log_a 2$ as $\log_a 128$. Is she correct? Explain your answer.

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COMMON CORE STATE STANDARDS FOR MATHEMATICS

LESSON NUMBER	PREREQUISITE SKILLS	SEEING STRUCTURE IN EXPRESSIONS	ARITHMETIC WITH POLYNOMIALS AND RATIONAL EXPRESSIONS	CREATING EQUATIONS	REASONING WITH EQUATIONS AND INEQUALITIES
1.1	✔				
1.2	✔				
1.3	✔				
1.4	✔				
1.5	✔				
1.6	✔				
1.7	✔				
1.8	✔				
1.9	✔				
1.10	✔				
1.11	✔				
1.12	✔				
1.13	✔				
1.14	✔				
1.15	✔				
1.16	✔				

LESSON NUMBER	PREREQUISITE SKILLS	SEEING STRUCTURE IN EXPRESSIONS	ARITHMETIC WITH POLYNOMIALS AND RATIONAL EXPRESSIONS	CREATING EQUATIONS	REASONING WITH EQUATIONS AND INEQUALITIES
1.17	✔				
1.19	✔				
1.20	✔				
1.21	✔				
1.22	✔				
2.1	✔				
2.2	✔				
2.3	✔				
2.4	✔				
2.5	✔				
2.6	✔				
2.7	✔				
2.8	✔				
2.9	✔				
2.10	✔				
2.11	✔				
2.12	✔				
2.13	✔				
2.14	✔				
2.15	✔				
2.16	✔				
2.17	✔				
2.18	✔				
2.19	✔				

LESSON NUMBER	PREREQUISITE SKILLS	SEEING STRUCTURE IN EXPRESSIONS	ARITHMETIC WITH POLYNOMIALS AND RATIONAL EXPRESSIONS	CREATING EQUATIONS	REASONING WITH EQUATIONS AND INEQUALITIES
3.1				✔	
3.2					✔
3.3					✔
3.4					✔
3.5		✔			✔
3.6					✔
3.7					✔
3.8		✔		✔	✔
3.9					✔
3.10					✔
3.11				✔	
3.12					✔
3.13				✔	
3.14					✔
3.15					✔
3.16		✔		✔	✔
3.17					✔
3.18					✔
3.19					✔
3.20					✔
3.21					✔
3.22					✔
3.23					✔
3.24					✔

LESSON NUMBER	PREREQUISITE SKILLS	SEEING STRUCTURE IN EXPRESSIONS	ARITHMETIC WITH POLYNOMIALS AND RATIONAL EXPRESSIONS	CREATING EQUATIONS	REASONING WITH EQUATIONS AND INEQUALITIES
4.1					✔
4.2					✔
4.3					✔
4.4					✔
4.5					✔
4.6				✔	✔
4.7					✔
4.8					✔
4.9					✔
4.10					✔
4.11					✔
4.12					✔
4.13				✔	✔
4.14				✔	✔
4.15					✔
4.16					✔
4.17					✔
4.18					✔
5.1		✔			
5.2		✔			
5.3			✔		
5.4			✔		
5.5		✔	✔		
5.6		✔	✔		

LESSON NUMBER	PREREQUISITE SKILLS	SEEING STRUCTURE IN EXPRESSIONS	ARITHMETIC WITH POLYNOMIALS AND RATIONAL EXPRESSIONS	CREATING EQUATIONS	REASONING WITH EQUATIONS AND INEQUALITIES
5.7		✔	✔		
5.8			✔		
5.9			✔		
5.10		✔			
5.11			✔		
5.12		✔			
5.13		✔	✔		
5.14		✔	✔		
5.15		✔	✔		
5.16		✔	✔		
5.17		✔	✔		
5.18		✔	✔		
5.19		✔	✔		
5.20		✔	✔	✔	✔
5.21		✔	✔	✔	✔
5.22		✔	✔	✔	✔
5.23		✔	✔		✔
6.1		✔			
6.2		✔			
6.3		✔			
6.4		✔	✔	✔	
6.5		✔	✔		
6.6			✔		
6.7		✔	✔		
6.8			✔		

LESSON NUMBER	PREREQUISITE SKILLS	SEEING STRUCTURE IN EXPRESSIONS	ARITHMETIC WITH POLYNOMIALS AND RATIONAL EXPRESSIONS	CREATING EQUATIONS	REASONING WITH EQUATIONS AND INEQUALITIES
6.9			✔		
6.10			✔		
6.11					✔
6.12					✔
6.13					✔
7.1		✔			
7.2		✔			
7.3		✔			
7.4		✔			
7.5		✔			
7.6		✔	✔		
7.7		✔	✔		
7.8		✔			
7.9		✔			
7.10		✔	✔		
8.1					✔
8.2					✔
8.3					✔
8.4					✔
8.5					✔
8.6					✔
8.7					✔
8.8					✔
8.9		✔	✔		✔

LESSON NUMBER	PREREQUISITE SKILLS	SEEING STRUCTURE IN EXPRESSIONS	ARITHMETIC WITH POLYNOMIALS AND RATIONAL EXPRESSIONS	CREATING EQUATIONS	REASONING WITH EQUATIONS AND INEQUALITIES
8.10		✔	✔	✔	✔
8.11		✔			✔
8.12				✔	✔
8.13					✔
8.14		✔			
8.15		✔			
8.16				✔	✔
8.17		✔			✔
8.18					✔
8.19				✔	✔
8.20					✔
8.21		✔			✔

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